

Straightened law for quantum isotropic Grassmannian $\mathrm{OGr}^+(5, 10)$

M.V. Movshev

Stony Brook University

Stony Brook, NY 11794-3651, USA

December 30, 2011

Abstract

Projective embedding of an isotropic Grassmannian $\mathrm{OGr}^+(5, 10)$ into projective space of spinor representation S can be characterized with a help of Γ -matrices by equations $\Gamma_{\alpha\beta}^i \lambda^\alpha \lambda^\beta = 0$. A polynomial function of degree N with values in S defines a map to $\mathrm{OGr}^+(5, 10)$ if its coefficients satisfy a $2N + 1$ quadratic equations. Algebra generated by coefficients of such polynomials is a coordinate ring of the quantum isotropic Grassmannian. We show that this ring is based on a lattice; its defining relations satisfy straightened law. This enables us to compute Poincaré series of the ring.

Mathematics Subject Classification(2010).

main 13F50, secondary 81R25, 81T30

Keywords. Straightened law, pure spinors, Delannoy numbers, Poincaré series

Contents

1	Introduction	2
2	Acknowledgments	7

3	Partial order on the basis of spinor representation S	8
4	Proof of the straightened law for algebra A	12
5	The Poincaré series of algebra A	20
6	Algebras of Richardson varieties	24
7	Properties of \hat{E} as partially ordered set	25
7.1	Description of the affine Weyl group $W(\hat{D}_5)$	27
7.2	Interpretation of \hat{E} in terms of the spinor representation $S[z, z^{-1}]$	28
8	Proof of the straightened law	29
9	Computation of Poincaré series $A_{\delta}^{\delta'}(z, q, t)$	36
9.1	Special cases	39
9.2	Delannoy polynomials	41
A	Some automorphisms of \hat{A}	42
B	Lists of obstructions	44

1 Introduction

Complex Isotropic Grassmannian $\text{OGr}^+(5, 10)$ or a space of pure spinors, as it is known in the physics literature, is a cornerstone of manifestly Poincaré covariant formulation of string theory [5]. A rigorous construction of the Hilbert space of string theory in the formalism of pure spinors remains a challenging problem (cf. [2]). Paper [1] investigates a simplified model with a quadric as the target. In [26] we proved rigorously most of the results of [1].

The paper [1] concerns certain algebra-geometric properties of the space of smooth maps $\text{Map}(S^1, Q) \subset \text{Map}(S^1, V^{2n})$ from a circle to a nondegenerate affine quadric Q in $2n$ -dimensional linear space V^{2n} . In [26] we observed that

analysis of [1] becomes rigorous if we replace $\text{Map}(S^1, Q)$ by the space of polynomial maps

$$\sum_{N \leq k \leq N'} \sum_{\mathbf{s} \in G_n} g_{\mathbf{s}}^k v_{\mathbf{s}} z^k$$

written in some basis $(v_{\mathbf{s}})$, $\mathbf{s} \in G_n = \{1, \dots, n, 1^*, \dots, n^*\}$ in V^{2n} , and then pass to a limit $N' \rightarrow \infty, N \rightarrow -\infty$. In this basis the $\text{SO}(2n)$ -invariant quadratic form q splits: $q = \sum_{i=1}^n x_i x_{i^*}$. An important technical observation, on which hinge all other results in [26], is that the algebra generated by $g_{\mathbf{s}}^k$ is the algebra with straightened law.

Recall (cf.[14]) that an algebra A over a ring R with straightened law is based on a directed graph F without oriented cycles. Generators τ^α of the algebra are labelled by vertices of the graph. The set of vertices is partially ordered: $\alpha \leq \beta$ if there is a directed path or a *chain*

$$\alpha \rightarrow \dots \rightarrow \beta \quad (1)$$

in F . A is an algebra with straightened law if:

1. *Standard monomials* $\tau^{\alpha_1} \dots \tau^{\alpha_n}$ labelled by $\alpha_1 \leq \dots \leq \alpha_n$ form a basis in A .
2. If α and β are incomparable and

$$\tau^\alpha \tau^\beta = \sum_i r_i \tau^{\gamma_{i,1}} \dots \tau^{\gamma_{i,k_i}} \quad (2)$$

where $0 \neq r_i \in R$, $\gamma_{i,1} \leq \dots \leq \gamma_{i,k_i}$ is the unique expression of $\tau^\alpha \tau^\beta$ in A as a linear combination of standard monomials, then $\gamma_{i,1} \leq \alpha, \beta$.

Throughout the paper the ground ring R is the field of complex numbers. The goal of the present paper is to establish straightened law for Quantum Isotropic Grassmannian in terminology of [29] or the space of Drinfeld's quasimaps of \mathbf{P}^1 to $\text{OGr}^+(5, 10)$ in terminology of [10].

The most straightforward way to define Quantum Isotropic Grassmannian is through Γ -matrices $\Gamma_{\alpha\beta}^{\mathbf{s}}$. They are defined as matrix coefficients of $\text{Spin}(10)$ -intertwiners

$$\text{Sym}^2 S \rightarrow V \quad (3)$$

in a basis

$$\{\theta_\alpha | \alpha \in E\} \quad (4)$$

of a spinor representation S and a basis

$$\{v_s | s \in G\} \quad G = G_5 = \{1, \dots, 5, 1^*, \dots, 5^*\} \quad (5)$$

in the defining representation $V = V^{10}$. Presently we need to know only that cardinality of E is equal to sixteen. By definition

$$\sum_{s \in G} \Gamma_{\alpha\beta}^s v_s = \Gamma(\theta_\alpha, \theta_\beta). \quad (6)$$

Generators of algebra $A_N^{N'}$, $N \leq N'$ of homogeneous functions on Quantum Isotropic Grassmannian can be arranged into generating functions

$$\lambda^\alpha(z) = \sum_{N \leq l \leq N'} \lambda^{\alpha^l} z^l, \quad \alpha^l \stackrel{\text{def}}{=} (\alpha, l) \in E \times \mathbb{Z} \stackrel{\text{def}}{=} \hat{E}. \quad (7)$$

The generating function of relations Γ^{s^l} is

$$\sum_l \Gamma^{s^l} z^l = \Gamma_{\alpha\beta}^i \lambda^\alpha(z) \lambda^\beta(z) \quad s^l \stackrel{\text{def}}{=} (s, l) \in G \times \mathbb{Z} \stackrel{\text{def}}{=} \hat{G} \quad (8)$$

Definition 1 *Quantum Isotropic Grassmannian is $\text{Proj}(A_N^{N'})$.*

The study of straightened law phenomenon is a part of Standard Monomial Theory (cf. [20]). It has been established for algebras of homogeneous functions on Schubert cells in partial flag spaces of semisimple groups. Straightened law has also been established [19] for intersections of Schubert varieties and opposite Schubert varieties (called Richardson varieties). Works [21], [22], and [23] generalize results of classical Standard Monomial Theory to symmetrizable Kac-Moody algebras. An algebra C of this loosely defined class enjoys the following properties:

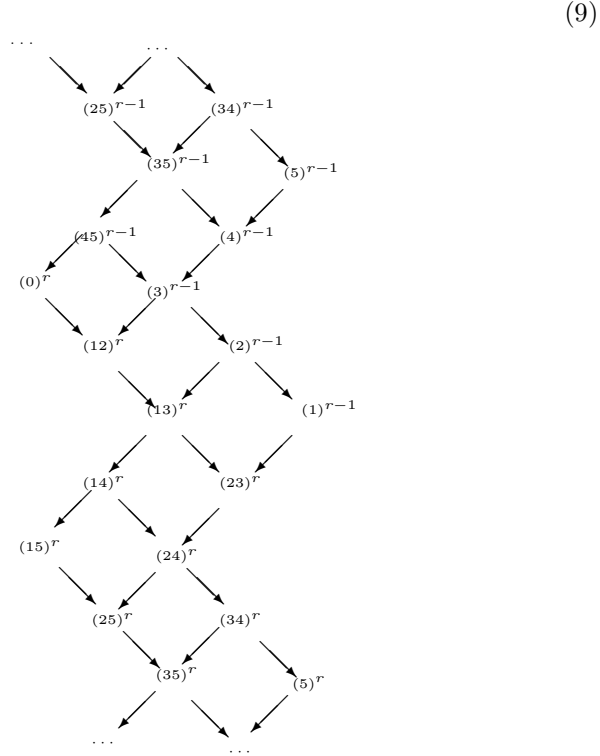
1. Straightened law holds.
2. Koszul property holds.

3. $\text{Spec}(C)$ is a reduced, irreducible, and normal scheme with Cohen-Macaulay singularities.

The objects that we are studying in this paper can be interpreted as Richardson varieties in a semi-infinite partial flag space of $\hat{\mathfrak{so}}_{10}$ (cf. [10]), which are closely related to the spaces discussed in [16] and [15]. It seems reasonable to expect that the suitably generalized technique of [19] would make it possible to prove the listed above package of properties for spaces of quasimaps to an arbitrary quotient G/P with a semisimple G and a parabolic P .

We however pursue a more modest goal to establish 1,2 and some of 3 for quantum $\text{OGr}^+(5,10)$. Our analysis relies on a set of identities between coefficients $\Gamma_{\alpha\beta}^s$ in relations (8). These Fierz identities are ubiquitous in gauge, gravity, and string theories.

The diagram \hat{E} is fundamental for Quantum Isotropic Grassmanians:



To make formulas more readable we denote elements of $\hat{\mathbb{E}}$ by Greek letters $\hat{\alpha}, \dots, \hat{\delta}$ with circumflex accent, elements of $\hat{\mathbb{G}}$ by similarly accented bold Roman letters $\hat{\mathbf{s}}, \hat{\mathbf{t}}$. Condition (1) defines a partial order on $\hat{\mathbb{E}}$. A segment $[\hat{\delta}, \hat{\delta}']$ is a subset $\{\hat{\alpha} \in \hat{\mathbb{E}} | \hat{\delta} \leq \hat{\alpha} \leq \hat{\delta}'\}$. A set of generators $\lambda^{\hat{\alpha}}$ of algebra $A_N^{N'}$ is labelled by $\hat{\alpha} \in [(0)^N, (1)^{N'}]$, $N \leq N'$. It is useful to consider a more general class of algebras $A_{\hat{\delta}}^{\hat{\delta}'}$ labelled by an interval $[\hat{\delta}, \hat{\delta}'] \subset \hat{\mathbb{E}}$. We shall refer to $Proj(A_{\hat{\delta}}^{\hat{\delta}'})$ as a (semi-infinite) *Richardson variety*.

In this paper we establish that straightened law for $A_{\hat{\delta}}^{\hat{\delta}'}$ (Proposition 32) can be derived from the Fierz identity (see Proposition 11). As an immediate corollary of straightened law we obtain Cohen-Macaulay property of $A_{\hat{\delta}}^{\hat{\delta}'}$ (Corollary 36), reducibility of $Spec(A_{\hat{\delta}}^{\hat{\delta}'})$ (Proposition 34), and Koszul property (Proposition 33).

Group of automorphisms of algebras $A_{\hat{\delta}}^{\hat{\delta}'}$ contains a maximal complex torus $\mathbf{T}^5 \subset Spin(10)$ and \mathbb{C}^\times (acts as a loop rotation). The characters of $\mathbf{T}^5 \times \mathbb{C}^\times$ action in graded components of $A_{\hat{\delta}}^{\hat{\delta}'}$ can be arranged into Poincaré series

$$A_{\hat{\delta}}^{\hat{\delta}'}(z, q, t), z \in \mathbf{T}^5, q \in \mathbb{C}^\times. \quad (10)$$

The formula (68) for $A_{\hat{\delta}}^{\hat{\delta}'}(z, q, t)$ can be obtained as a corollary of straightened law. Let J be equal to

$$\{\dots, \hat{\delta}_0, \hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_4, \hat{\delta}_5, \dots\} = \{\dots, (15), (5), (0)^1, (1), (15)^1, (5)^1, \dots\} \quad (11)$$

The functions $B_r(t) = A_{(0)}^{\hat{\delta}_r}(1, 1, t)$, $r \geq 0$ satisfy recursion (see Section 9.1)

$$\begin{aligned} B_r(t) &= \frac{1+t}{(1-t)^2} B_{r-1}(t) + \frac{t}{(1-t)^4} B_{r-2}(t) \\ B_0(t) &= \frac{1}{(1-t)^5}, B_1(t) = \frac{1+t}{(1-t)^7} \end{aligned} \quad (12)$$

This gives an effective tool for computation of B_r and is closely related to recursion for Delannoy numbers (cf.[3]). In fact, author came across these formulas during computer experiments with $A_{(0)}^{\hat{\delta}_r}(1, 1, t)$ and one of the goals of the present paper was an explanation of these findings.

Several interesting topics have been left outside of the scope of present paper: firstly, the analysis of limiting characters $A_{\hat{\delta}}^{\hat{\delta}'}(z, q, t)$, $\hat{\delta} \rightarrow \infty$; secondly, the

construction of the Hilbert space of $\beta\gamma$ -system on pure spinor. These will await a future publication.

The main results of the present paper are Corollaries 32–33, Proposition 34, 35 and formulas (68), (12). Propositions 27, 29 are technically central in this paper.

The paper is organized as follows. The principal construction, the Hasse diagram based on set of weights E of spinor representation, is introduced in Section 3. For readers convenience we sketch the proof of straightened law in the case of ordinary pure spinors with an emphasis on Fierz identities. The proof is done in Section 4. Combinatorial derivation of the character of $OGr^+(5, 10)$ is given in Section 5. Algebras of Richardson varieties are defined in Section 6. Section 7 contains some preliminary results on the structure of a partially ordered set \hat{E} . Straightened law for quantum $OGr^+(5, 10)$ and for $Spec(A_{\delta}^{\hat{\delta}'})$ are proved in Section 8. Cohen-Macaulay property, dimension, and depth of these varieties are also established Section 8. A formula for character $A_{\delta}^{\hat{\delta}'}(z, q, t)$ is established in Section 9. Note that Sections 9 and 8 are logical continuation of Sections 4 and 5. Section 9.2 contains relevant facts about Delannoy numbers. Appendix contains some technical information. In Appendix A we justify existence of automorphisms that are needed in the proof of straightened law. Appendix B contains a list of obstructions that is used in the proof of the main result.

2 Acknowledgments

The author would like to thank A.S. Schwarz for useful comments. Parts of this work has been written during the authors stay at IHES and Max Plank Institute for Mathematics (Bonn). The author is grateful to these institutions for excellent working conditions.

3 Partial order on the basis of spinor representation S

In this section we define a partial order in the basis of 16-dimensional spinor representation S of the complex algebraic group $\text{Spin}(10)$, the group of type D_5 in Cartan classification. This order is fundamental for the definition of the straightened law.

Let V be a fundamental complex 10-dimensional vector representation of complex $\text{SO}(10)$. It carries an $\text{SO}(10)$ -invariant inner product (\cdot, \cdot) , the polarization of the quadratic form q . We choose a decomposition $W + W' = V$. Subspaces W and W' are isotropic and the inner product satisfies

$$(v_i, v_{j*}) = \delta_{ij}. \quad (13)$$

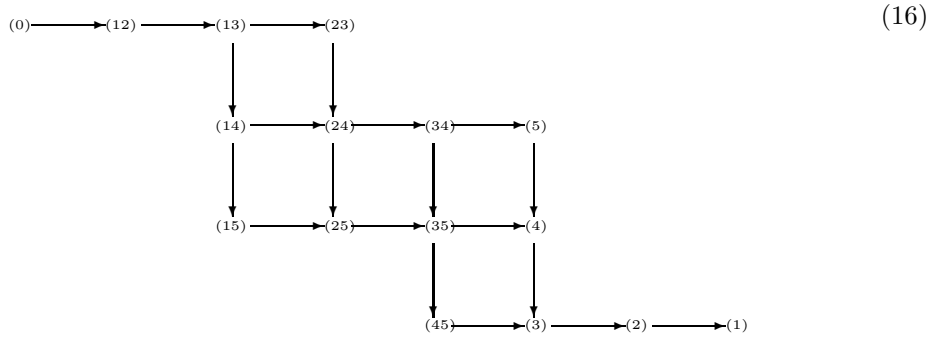
Recall (see [11] for details) that the spinor representation admits a construction of a fermionic Fock space. The spinor representation S can be identified with the direct sum

$$S = \bigoplus_{i \geq 0} \Lambda^{2i} W. \quad (14)$$

Tensors $\theta_{(ij)} = v_i \wedge v_j$, $\theta_{(k)} = v_1 \wedge \cdots \wedge \hat{v}_k \cdots \wedge v_5$ and a constant $1 = \theta_{(0)}$ define a bases in S . We set (cf. 4)

$$E = \{(0), (ij), (k) | 1 \leq i < j \leq 5, 1 \leq k \leq 5\} \quad (15)$$

By abuse of notation we shall denote by E the following Hasse diagram, whose vertices are labelled by weights of S :



A spinor $\theta \in S$ can be written as a sum

$$\theta = \lambda \theta_{(0)} + \sum_{i < j} w_{ij} \theta_{(ij)} + \sum_{k=1}^5 p_k \theta_{(k)} = \sum_{\alpha \in E} \lambda^\alpha \theta_\alpha$$

The Lie algebra $\mathfrak{gl}_5 \subset \mathfrak{so}_{10}$ acts tautologically on W , via contragradient representation on W' and diagonally on $W + W'$. Vectors (θ_α) define a weight basis for the algebra of diagonal matrices

$$\mathfrak{h} \subset \mathfrak{gl}_5 \subset \mathfrak{so}_{10} \quad (17)$$

The action of the subgroup of diagonal matrices $\tilde{H} \subset \widetilde{\text{GL}}(5) \subset \text{Spin}(10)$ (\sim stands for two-sheeted cover) on the weight vectors is

$$\begin{aligned} \rho(z) \theta_{(0)} &= \det^{-\frac{1}{2}}(z) 1 \stackrel{\text{def}}{=} e_{(0)}(z) 1 \\ \rho(z) \theta_{(ij)} &= \det^{-\frac{1}{2}}(z) z_i z_j \theta_{(ij)} \stackrel{\text{def}}{=} e_{(ij)}(z) \theta_{(ij)} \\ \rho(z) \theta_{(k)} &= \det^{\frac{1}{2}}(z) z_k^{-1} \theta_{(k)} \stackrel{\text{def}}{=} e_{(k)}(z) \theta_{(k)} \end{aligned} \quad (18)$$

In view of this we can identify E with the set of H -weights in S .

Following Section 7 in [20] we identify the Weyl group $W(D_5)$ with a subgroup of a semidirect product $S_5 \ltimes (\mathbb{Z}_2)^5$. The symmetric group acts by permutation of entries of the array $(\epsilon_1, \dots, \epsilon_5) \in (\mathbb{Z}_2)^5$. The group $W(D_5)$ is a semidirect product $S_5 \ltimes N$, with $N \cong (\mathbb{Z}_2)^4$ being a kernel of the map $(\mathbb{Z}_2)^5 \rightarrow \mathbb{Z}_2$

$$(\epsilon_1, \dots, \epsilon_5) \rightarrow \sum_{i=1}^5 \epsilon_i. \quad (19)$$

Representation S is *minuscule*, i.e. $W(D_5)$ acts transitively on E ([20]). The $W(D_5)$ orbit of the highest vector identifies with the coset $S_5 \ltimes N / S_5$. The elements

$$(0, 0, 0, 0, 0), (0 \dots, \overset{i}{1}, \dots, 0, \dots, \overset{j}{1}, \dots, 0), (1, \dots, \overset{k}{0}, \dots, 1)$$

in $S_5 \ltimes N / S_5 \cong (\mathbb{Z}_2)^4 \subset (\mathbb{Z}_2)^5$ correspond to elements $(0), (ij), (k)$ in E

Let $\sigma_{ij} \in S_5$ be a permutation, $r_{\mathbf{m}}$ be an operator of multiplication on $\mathbf{m} \in N$. We shall be using the following set of Coxeter generators $s_1 = \sigma_{12}, s_2 = r_{(1,1,0,0,0)}, s_3 = \sigma_{23}, s_4 = \sigma_{34}, s_5 = \sigma_{45}$, which can be arranged into the Coxeter graph (see [17] for details):

Two vertices $a \neq b$ in X are connected by an edge in $Q(X, s_1, \dots, s_k)$ if $b = s_i a$ for some s_i . In case $b = s_i a$ for two values of i we still have one connecting edge.

Proposition 4 *The graph $Q(S_5 \ltimes N/S_5, s_1, \dots, s_5)$ is isomorphic to a non-directed version of (16).*

Proof. Straightforward exercise. ■

Left multiplication on v_i and contraction with v_j^* define operators in ΛW . By abuse of notations we denote these operators by v_i, v_j^* . Operators $v_i v_j, v_i v_j^*, v_i^* v_j^*, 1$ satisfy commutation relation of $\mathfrak{so}_{10} \times \mathbb{C}$ (cf. [12]). Operators $v_i v_i^*, 1, i = 1, \dots, 5$ define the action of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_5 \subset \mathfrak{so}_{10} \times \mathbb{C}$ in S . Operators in S corresponding to simple positive root vectors $e_i, i = 1, \dots, 5$ in \mathfrak{so}_{10} are

$$R_2^+ x = v_1 v_2 x \quad (22)$$

and

$$R_i^+ x = v_i v_{i-1}^* x, \quad i = 1, 3, 4, 5. \quad (23)$$

We connect $\alpha, \alpha' \in E$ by a directed edge if $R_i^+ \theta_\alpha$ is proportional to $\theta_{\alpha'}$ for some i . The choice of \mathfrak{h} and $\{e_i\}$ defines a triangular decomposition

$$\mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+ = \mathfrak{so}_{10} \quad (24)$$

Proposition 5 *Directed graph obtained this way is isomorphic to E .*

Proof. Direct inspection. ■

In the following we shall not make a distinction between directed graphs with without directed cycles and partially ordered sets they define.

The points of the affine cone defined by equation

$$\Gamma^{\mathbf{s}} \stackrel{\text{def}}{=} \sum_{\alpha, \beta \in E} \Gamma_{\alpha\beta}^{\mathbf{s}} \lambda^\alpha \lambda^\beta = 0 \quad \mathbf{s} \in G \quad (25)$$

are called pure spinors. It is known [11] that $Proj(A) = OGr^+(5, 10)$ where

$$A = P/(\Gamma^{\mathbf{s}}) \quad \mathbf{s} \in G$$

According to [5] and [13] Γ^i and Γ^{i*} are respectively

$$\begin{aligned}\lambda p_i - \text{Pf}_i(w) &= 0 \quad i = 1, \dots, 5 \\ wp &= 0.\end{aligned}\tag{26}$$

Last line contains five equations, written in a matrix form. The function $\text{Pf}_i(w)$ is the pfaffian of the complement of i -th row and column in w .

4 Proof of the straightened law for algebra A

Formally this section contains no new results. The goal we pursue here is to familiarize the reader with simplified version of the proof of the main statements in a well studied setting with an emphasis on specifics of a spinor representation.

Proposition 6 [20]

- i. Each of the equations (26) and their expanded version (28) contains a unique clutter.*
- ii. More precisely equations have the form*

$$\lambda^\alpha \lambda^{\alpha'} \pm \lambda^{\alpha \vee \alpha'} \lambda^{\alpha \wedge \alpha'} = \sum_{\gamma < \alpha \wedge \alpha', \gamma' > \alpha \vee \alpha'} \pm \lambda^\gamma \lambda^{\gamma'} \tag{27}$$

Proof. We have already mentioned that several proofs that work in much more general context are known (see Introduction), we give one proof here with a purpose of generalization in Proposition 27.

Quadrics (26) can be written in the expanded form:

$$\begin{aligned}
\Gamma^1 &= \lambda p_1 + \underbrace{w_{2,5}w_{3,4}} - \overbrace{w_{2,4}w_{3,5}} + w_{2,3}w_{4,5} \sim 0 \\
\Gamma^2 &= -\lambda p_2 - \underbrace{w_{1,5}w_{3,4}} + \overbrace{w_{1,4}w_{3,5}} - w_{1,3}w_{4,5} \sim 0 \\
\Gamma^3 &= \lambda p_3 + \underbrace{w_{1,5}w_{2,4}} - \overbrace{w_{1,4}w_{2,5}} + w_{1,2}w_{4,5} \sim 0 \\
\Gamma^4 &= -\lambda p_4 - \underbrace{w_{1,5}w_{2,3}} + \overbrace{w_{1,3}w_{2,5}} - w_{1,2}w_{3,5} \sim 0 \\
\Gamma^5 &= \lambda p_5 + \underbrace{w_{1,4}w_{2,3}} - \overbrace{w_{1,3}w_{2,4}} + w_{1,2}w_{3,4} \sim 0 \\
\Gamma^{1*} &= -p_2w_{1,2} + p_3w_{1,3} - \overbrace{p_4w_{1,4}} + \underbrace{p_5w_{1,5}} \sim 0 \\
\Gamma^{2*} &= -p_1w_{1,2} + p_3w_{2,3} - \overbrace{p_4w_{2,4}} + \underbrace{p_5w_{2,5}} \sim 0 \\
\Gamma^{3*} &= -p_1w_{1,3} + p_2w_{2,3} - \overbrace{p_4w_{3,4}} + \underbrace{p_5w_{3,5}} \sim 0 \\
\Gamma^{4*} &= -p_1w_{1,4} + p_2w_{2,4} - \overbrace{p_3w_{3,4}} + \underbrace{p_5w_{4,5}} \sim 0 \\
\Gamma^{5*} &= -p_1w_{1,5} + p_2w_{2,5} - \overbrace{p_3w_{3,5}} + \underbrace{p_4w_{4,5}} \sim 0
\end{aligned} \tag{28}$$

A brute force approach to the proof requires verification conditions of the proposition for all equations (28). Monomials marked by underbracket $\underline{}$ are clutters. Monomials $\lambda^{\alpha \wedge \beta} \lambda^{\alpha \vee \beta}$ are marked by overbracket $\overline{}$. We leave verifications to the reader.

There is a less computational but lengthier proof, based on representation theory. We include it here as a prototype of the proof in the affine case.

Spinors decompose under $\mathfrak{so}_8 \subset \mathfrak{so}_{10}$ into a direct sum

$$S = S^+ + S^- \tag{29}$$

of eight-dimensional spinor representations of opposite chiralities (see e.g. [30]). We choose Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}_8 \times \mathfrak{so}_2 \subset \mathfrak{so}_{10}$. Γ maps weight vectors $\theta^\alpha \theta^{\alpha'}$ in the space of symmetric tensors $\text{Sym}^2 S$ to weight vectors $\Gamma(\theta^\alpha, \theta^{\alpha'}) \in V$. A pair of vectors $v_+ = v_i, v_- = v_{i*} \in V$ (5) is invariant with respect to some $\mathfrak{so}_8 \subset \mathfrak{so}_{10}$ action and get scaled under action of $\mathfrak{so}_2 \subset \mathfrak{so}_{10}$. The remaining $\{v_j, v_{j*} | j \neq i\}$ span eight-dimensional vector representation of \mathfrak{so}_8 . Quadratic functions $\Gamma^+ = \Gamma^i, \Gamma^- = \Gamma^{i*}$ define nonzero \mathfrak{so}_8 -invariant inner products $(\cdot, \cdot)_\pm$

on S^\pm . Note that \mathfrak{so}_2 -scaling of v_\pm insures (cf. 6) that $\Gamma^+|_{S^-} = \Gamma^-|_{S^+} = 0$. Γ^\pm are unique up to a factor (see [30]).

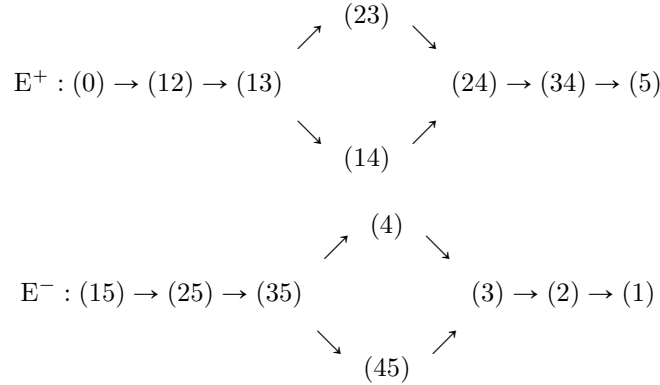
Let

$$\mathfrak{n}'_- + \mathfrak{h} + \mathfrak{n}'_+ = (\mathfrak{n}_+ \cap \mathfrak{so}_8) + (\mathfrak{h} \cap \mathfrak{so}_8) + (\mathfrak{n}_- \cap \mathfrak{so}_8) \quad (30)$$

the decomposition of \mathfrak{so}_8 compatible with (24). Let $\gamma_{\alpha, \alpha'}$ be a path in E that connects α and α' . It is easy to see that such path exists if and only if there is an element u in universal enveloping algebra $U(\mathfrak{n}_+)$ such that $u\theta^\alpha$ is proportional to $\theta^{\alpha'}$.

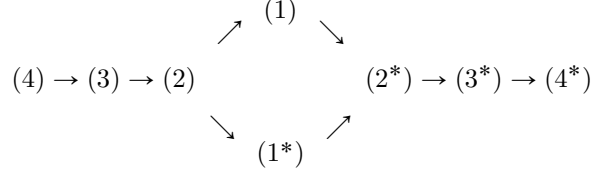
We decompose E into a union $E^+ \cup E^-$, with $E^\pm = \{\alpha \in E | \theta^\alpha \in S^\pm\}$. A poset structure on E^\pm is defined as follows: a directed path $\gamma_{\alpha, \alpha'}$ connects $\alpha, \alpha' \in E^\pm$ if there $u' \in U(\mathfrak{n}'_+)$ with $u\theta^\alpha \sim \theta^{\alpha'}$. In this case we define α to be less or equal to α' .

Simple positive root vectors in \mathfrak{so}_8 with respect to decomposition (30) are $v_2v_1^*, v_3v_2^*, v_4v_3^*, v_1v_2$ if $v_+ = v_5, v_- = v_5^*$. Their action on weight vectors in $S^\pm \subset S$, which are also \mathfrak{so}_{10} weight vectors, produce the following subdiagram in E



One of the triality automorphisms of \mathfrak{so}_8 that conjugates S^\pm with the defining representation V^8 preserves decomposition (30), shuffles simple roots, and conjugate weight vectors and quadratic forms. This is why Hasse diagrams of S^\pm and V^8 are the same. Hasse diagrams of defining representations of \mathfrak{so}_{2n}

has been studied in [26]. When $n = 4$ such diagram is



Quadratic form on $V^8 \sum_{i=1}^4 x_i x^{i*}$ in the basis (v_s) contains a unique clutter $x_1 x^{1*}$. Therefore the conjugated Γ^+ contains a unique clutter $\lambda^{14} \lambda^{23}$ and the conjugated Γ^- contains a unique clutter $\lambda^4 \lambda^{45}$.

The group $W(D_5)$ acts transitively on weights of V . By Proposition (2) all clutters belong to one $W(D_5)$ -orbit. The Weyl group shuffles the quadrics Γ^s . The w -transformed quadric $\Gamma^s = (\Gamma^+)^w$, $w \in W(D_5)$ contains any given clutter. Ten - the number of quadrics coincides with the number of clutters. A quadric can contain at least one clutter, because an E-clutter is an E^\pm -clutter, which is unique as we saw above. Thus, each Γ^s contains a unique E-clutter.

■

Corollary 7 *There is a one-to-one correspondence between a set of clutters $(\alpha, \alpha') \in M \subset E \times E$ and quadrics Γ^s , $s \in G$.*

Poincaré series $A(t) = \sum_{n \geq 0} \dim A_k t^k$ has been determined in [6] using fixed point technique and in [13] using method of resolutions. Our method relies on Gröbner basis technique. It admits a straightforward generalization to the algebras $A_{\hat{\delta}}^{\hat{\delta}'}$ (see Section 8).

Relations (27) provide us with a combinatorial method to compute $A(t)$. The method amounts to constructing a basis in $A_k \subset A$. Consequently $\dim A_k$ is exactly the number of elements in the basis. Monomials $\lambda^n = \prod_{\alpha \in E} (\lambda^\alpha)^{n_\alpha}$, $\deg \lambda^n = \overline{n} = k$ define a basis in P , but because of relations in A some inevitably become redundant in A_k . Our goal is to construct a subset $B_k \subset \{\lambda^n | \overline{n} = k\}$, which becomes a basis in A_k . Any monomial λ^n , because of the relations (27), is a linear combination of standard monomials (see Section 1).

We denote by X_k the set of standard monomials of degree k . The process of elimination of clutters using relations is called reduction. We associate a reduction with each Γ^s in (28).

The set X_k might not be a basis. Some monomials like $p_5 w_{2,5} w_{3,4}$, which are called *obstructions*, admit several reductions: we can apply reduction Γ^{2*} to $p_5 w_{2,5}$ or Γ^1 to $w_{2,5} w_{3,4}$.

We shall use term obstruction also for a set of unordered triples $(\alpha, \beta, \gamma) \in \text{Sym}^3 E$ such that $(\alpha, \beta), (\alpha, \gamma) \in M$. They encode obstructing monomials $\lambda^\alpha \lambda^\beta \lambda^\gamma$. For technical purposes we need to modify slightly this definition.

Definition 8 *An (unordered) list $(\gamma, (\alpha, \beta))$ of elements in a poset E is called a noncommutative obstruction if (α, β) is a clutter and if (α, γ) or (β, γ) is a clutter. Thus noncommutative obstructions come in pairs:*

$$\{(\gamma, (\alpha, \beta)), (\alpha, (\beta, \gamma))\} \text{ or } \{(\gamma, (\alpha, \beta)), (\beta, (\alpha, \gamma))\}$$

By abuse of notations we shall also call a monomial $\lambda^\gamma \otimes \lambda^\alpha \lambda^\beta$ a noncommutative obstruction.

If $(\gamma, (\alpha, \beta))$ is a noncommutative obstruction, then (γ, α, β) is an ordinary obstruction. A list (75) of pairs noncommutative obstructions in E is given in Appendix B.

It is conceivable that different reductions of an obstruction result in distinct non reducible expressions. It would mean that we need to further reduce the set X_k . Luckily, this does not happen in our case. There is a technology of Gröbner bases (see [25] for introduction and references), which has been specifically designed to deal with such issues. In order to apply it we refine our partial order on E to a strict total order:

$$\begin{aligned} (0) &< (12) < (13) < (14) < (23) < (15) < (24) < (25) < \\ &< (34) < (35) < (5) < (45) < (4) < (3) < (2) < (1) \end{aligned} \tag{31}$$

We extend this order to degree lexicographic order in the monomial basis in P . The tip $T(f) = lc(f) \lambda^n$ is the maximal monomial of $f \in P$, $lc(f)$ is the

leading coefficient. We write $f > g$ for polynomials f, g with $T(f) > T(g)$. The order is chosen so that reductions decrease it:

$$T(\Gamma^{\mathbf{s}}) > T(\Gamma^{\mathbf{s}}) - \Gamma^{\mathbf{s}}$$

The "Diamond Lemma" [4] guarantees that if results of any two reduction of all (cubic) obstructions agree, then X_k is a basis in A_k . To be precise we define $T(f, g)$ to be the least common multiple (*lcm*) of $T(f)$ and $T(g)$.

Definition 9 *The polynomial*

$$S(f, g) = lc(f) \frac{T(f, g)}{T(g)} g - lc(g) \frac{T(f, g)}{T(f)} f$$

is called S-polynomial of f, g .

Let $\lambda^{\mathbf{n}}$ be some cubic obstruction, $\Gamma^{\mathbf{s}}, \Gamma^{\mathbf{s}'}$ be a pair of relations in (28) such that $lcm(T(\Gamma^{\mathbf{s}}), T(\Gamma^{\mathbf{s}'})) = \lambda^{\mathbf{n}}$, then $S(\Gamma^{\mathbf{s}}, \Gamma^{\mathbf{s}'})$ contains no multiple of $\lambda^{\mathbf{n}}$: it has been canceled in the subtraction.

Proposition 10 *If for any pair of generating relations $\Gamma^{\mathbf{s}}, \Gamma^{\mathbf{s}'} \in I \subset P$ ($\mathbf{s}, \mathbf{s}' \in G$) with $T(\Gamma^{\mathbf{s}}, \Gamma^{\mathbf{s}'}) \neq T(\Gamma^{\mathbf{s}})T(\Gamma^{\mathbf{s}'})$ the S-polynomials are*

$$S(\Gamma^{\mathbf{s}}, \Gamma^{\mathbf{s}'}) = \sum_{\alpha \in E} \sum_{\mathbf{r} \in G} c_{\alpha, \mathbf{r}} \lambda^{\alpha} \Gamma^{\mathbf{r}} \quad (32)$$

with no obstructions among monomials in $\lambda^{\alpha} \Gamma^{\mathbf{r}}$, then the set X_k is a basis in A_k .

Proof. This is a trivial restatement of Buchberger algorithm for a commutative Gröbner basis ([25]) adapted to our needs. More precisely this is a statement that the algorithm terminates at the first iteration. ■

The inner product (13) can be used to raise and to lower the \mathbf{s} -index in the tensor $\Gamma_{\alpha\beta}^{\mathbf{s}}$.

Proposition 11 *Quadratic functions $\Gamma^{\mathbf{s}}$ satisfy*

$$I_{\alpha} = \sum_{\mathbf{s} \in G} \Gamma_{\alpha\beta}^{\mathbf{s}} \lambda^{\beta} \Gamma_{\mathbf{s}} = 0 \quad (33)$$

Proof. This is equivalent to Fierz identity

$$\Gamma_{\alpha(\beta}^s \Gamma_{\gamma\delta)}^s = 0.$$

(\cdot) stands for symmetrization.

Theory of invariants gives a different outlook on this identity. It is not hard to see that $I_\alpha d\lambda^\alpha$ is a differential of $I(\lambda) = \Gamma^s \Gamma_s = \sum_{i=1}^5 \Gamma^i \Gamma^{i*}$. The function $I(\lambda)$ is by construction a $\text{Spin}(10)$ -invariant. By Igusa's classification of spinors in ten dimensions [18] S contains a dense orbit. This implies that $I(\lambda) \equiv I(0) = 0$, hence $I_\alpha = 0$. ■

Proposition 12 *Fix the order (31) on generators of P . Then assumptions of Proposition 10 are satisfied for relations (28).*

Proof. The algebra P coincides with symmetric algebra $\text{Sym } S^*$. Collection of Fierz identities describes a basis in the kernel of the multiplication map $S^* \otimes V^* \rightarrow \text{Sym}^3 S^*$:

$$\lambda^\alpha x_s \rightarrow \lambda^\alpha \Gamma^s \tag{34}$$

The kernel as a representation of \mathfrak{so}_{10} is isomorphic to S (e.g. [30]). Its weights are opposite to weights of S^* . We use these weights to label individual Fierz identities:

$$\begin{aligned}
h_{(0)} &= p_1 x_1 * - p_2 x_2 * + p_3 x_3 * - \underline{p_4 x_4 * + p_5 x_5 *} \\
h_{(12)} &= \underline{w_{4,5} x_3 *} - \underline{w_{3,5} x_4 *} + w_{3,4} x_5 * - p_2 x_1 - p_1 x_2 \\
h_{(13)} &= - \underline{w_{4,5} x_2 *} + \underline{w_{2,5} x_4 *} - w_{2,4} x_5 * + p_3 x_1 - p_1 x_3 \\
h_{(23)} &= \underline{w_{4,5} x_1 *} - \underline{w_{1,5} x_4 *} + w_{1,4} x_5 * + p_3 x_2 + p_2 x_3 \\
h_{(14)} &= \underline{w_{3,5} x_2 *} - \underline{w_{2,5} x_3 *} + w_{2,3} x_5 * - p_4 x_1 - p_1 x_4 \\
h_{(24)} &= - \underline{w_{3,5} x_1 *} + \underline{w_{1,5} x_3 *} - w_{1,3} x_5 * - p_4 x_2 + p_2 x_4 \\
h_{(15)} &= - \underline{w_{3,4} x_2 *} + w_{2,4} x_3 * - w_{2,3} x_4 * + \underline{p_5 x_1} - p_1 x_5 \\
h_{(34)} &= \underline{w_{2,5} x_1 *} - \underline{w_{1,5} x_2 *} + w_{1,2} x_5 * - p_4 x_3 - p_3 x_4 \\
h_{(25)} &= \underline{w_{3,4} x_1 *} - w_{1,4} x_3 * + w_{1,3} x_4 * + \underline{p_5 x_2} + p_2 x_5 \\
h_{(5)} &= \lambda x_5 * + \underline{w_{1,5} x_1} + \underline{w_{2,5} x_2} + w_{3,5} x_3 + w_{4,5} x_4 \\
h_{(35)} &= - \underline{w_{2,4} x_1 *} + w_{1,4} x_2 * - w_{1,2} x_4 * + \underline{p_5 x_3} - p_3 x_5 \\
h_{(4)} &= - \lambda x_4 * - w_{1,4} x_1 - \underline{w_{2,4} x_2} - \underline{w_{3,4} x_3} + w_{4,5} x_5 \\
h_{(45)} &= \underline{w_{2,3} x_1 *} - w_{1,3} x_2 * + w_{1,2} x_3 * + \underline{p_5 x_4} + p_4 x_5 \\
h_{(3)} &= \lambda x_3 * + w_{1,3} x_1 + \underline{w_{2,3} x_2} - \underline{w_{3,4} x_4} - w_{3,5} x_5 \\
h_{(2)} &= - \lambda x_2 * - w_{1,2} x_1 + \underline{w_{2,3} x_3} + \underline{w_{2,4} x_4} + w_{2,5} x_5 \\
h_{(1)} &= \lambda x_1 * - w_{1,2} x_2 - w_{1,3} x_3 - \underline{w_{1,4} x_4} - \underline{w_{1,5} x_5}
\end{aligned} \tag{35}$$

Monomials $T(\Gamma^s, \Gamma^{s'})$ such that $T(\Gamma^s, \Gamma^{s'}) \neq T(\Gamma^s)T(\Gamma^{s'})$ are degree three obstructions. Each h_α upon substitution (34) contains precisely a one pair of relations $\Gamma^s, \Gamma^{s'}$ (the corresponding terms are underlined) with $T(\Gamma^s, \Gamma^{s'}) \neq T(\Gamma^s)T(\Gamma^{s'})$. The reader can check that underscoring terms contain the same h_α -dependent obstruction monomial. For example, $p_4 \Gamma^{4*}$ and $p_5 \Gamma^{5*}$ in the first identity contain $p_4 p_5 w_{45}$. These monomials have opposite signs and cancel each other because of that. No other term in h_α contains an obstruction. A straightforward analysis of the lattice E reveals that there are sixteen obstructions of degree three (see Table 75) and each of them appears as monomial in one of the underlined terms.

We use these identities to rewrite S -polynomials in the form (63)

■

Corollary 13 (cf.[20]) *A is an algebra with straightened law. In particular standard monomials form a \mathbf{T}^5 weight basis.*

Corollary 14 (cf. [28] [9] [7]) *Fix the order (31) on generators of P . Then relations (28) form a quadratic Gröbner basis (see [27] for details) in the ideal I . A is a Koszul algebra*

Proof. Koszul property is a corollary (see [27]) of existence of quadratic Gröbner basis. ■

We define algebras $A_\delta^{\delta'}$ as follows. An ideal $J_\delta^{\delta'} \subset A$ is generated by $\{\lambda^\alpha | \alpha \in E \setminus [\delta, \delta']\}$, then

$$A_\delta^{\delta'} = A/J_\delta^{\delta'}.$$

Projective spectrum $\text{Proj}(A_\delta^{\delta'})$ is called a Richardson variety [19]. In particular $\text{Proj}(A_{(0)}^\delta)$ a Schubert variety; $\text{Proj}(A_\delta^{(1)})$ is the opposite Schubert variety.

Remark 15 *Relations in $A_\delta^{\delta'}$ are one-to-one correspondence with clutters in $[\delta, \delta']$. Statements of Corollaries 13 14 are valid for algebras $A_\delta^{\delta'}$ ([20], [28] [9] [7]). Standard monomials $\lambda^{\alpha_1} \dots \lambda^{\alpha_n}$ $\alpha_1 \leq \dots \leq \alpha_n \in [\delta, \delta']$ form a basis in $A_\delta^{\delta'}$.*

Example 16 *Algebra $A_{(0)}^{(15)}$ coincides with $\mathbb{C}[\lambda^0, \lambda^{12}, \lambda^{13}, \lambda^{14}, \lambda^{15}]$ because $[(0), (15)]$ contains no clutters (cf. 16). The map of algebras $A \rightarrow A_{(0)}^{(15)}$ encodes an embedding $\mathbf{P}^4 \subset \text{OGr}^+(5, 10)$.*

Example 17 *Algebra $A_{(0)}^{(5)}$ coincides with*

$$\mathbb{C}[\lambda^0, \lambda^{12}, \lambda^{13}, \lambda^{14}, \lambda^{24}, \lambda^{34}, \lambda^5] / (\lambda^0 \lambda^5 + \lambda^{14} \lambda^{23} - \lambda^{13} \lambda^{2,4} + \lambda^{12} \lambda^{34})$$

because $[(0), (5)]$ contains one clutter (cf. 16). This clutter is associated with relation Γ^5 (28). The polynomial algebra is isomorphic to symmetric algebra of eight-dimensional spinors S^+ . Relation defines quadratic form on S^+ , which appears in the proof of Proposition 6. Geometrically projection $A \rightarrow A_{(0)}^{(5)}$ defines an embedding of a quadric Q^6 into $\text{OGr}^+(5, 10)$.

5 The Poincaré series of algebra A

Poincaré series $A(t)$ of algebra A depends only on the combinatorics of the poset E . We choose to define generating function $C(F)(t)$ of a finite poset F , which in

case of E coincides with $A(t)$. In this section we also establish relations between $C(F)(t)$ and $C(F')(t)$ for some simple subposets $F' \subset F$. These relations define recursive relations between $A_\delta^{\delta'}(t)$ for different δ and δ' .

To define a generating function of a poset F we introduce a polynomial algebra $\mathbb{C}[e_\alpha]$, whose generators are labelled by elements of F . The generating function $C(F) \in \mathbb{C}[e_\alpha][[t]]$ does a weighted counting of chains

$$C_k(F) = \{\alpha_1 \leq \dots \leq \alpha_k \mid \alpha_i \in F\}$$

by the formula

$$C(F)(t) = \sum_{k \geq 0} \sum_{\{\alpha_1 \leq \dots \leq \alpha_k\} \in C_k(F)} e_{\alpha_1} \dots e_{\alpha_k} t^k \quad (36)$$

The chains in E are one-to-one correspondence with standard monomials, which by Corollary 13 form a \mathbf{T}^5 -basis in A . Poincaré series $A(z, t)$ (10) is some specialization $C(E)(t)$. The \mathbf{T}^5 -action on λ^α is multiplication on the weight $e_\alpha = e_\alpha(z) = e_\alpha(z_1, \dots, z_5)$ (18). Under this specialization the coefficient of t^k in (36) is the scaling factor of \mathbf{T}^5 -action on $\lambda^{\alpha_1} \dots \lambda^{\alpha_k}$.

More generally $A_\delta^{\delta'}(z, t)$ is equal to $C([\delta, \delta'])(t)$ under this specialization of e_α .

A bit of terminology: F is a *convenient* lattice if $[\delta, \delta']$ is a union of at most two intervals

$$[\delta, \delta'] = [\delta, \alpha] \cup [\delta, \alpha'] \quad (37)$$

If $[\delta, \delta'] = [\delta, \alpha]$ for some α , then δ' is called a *tail* of $[\delta, \delta']$. We say that a convenient lattice F is *narrow* if for any $\delta' \in F$, as in (37), such that $\alpha \neq \alpha'$ either α or α' is a tail.

Direct inspection of the diagram (16) shows that lattice E is convenient and narrow.

Proposition 18 *Let F be convenient and narrow finite lattice. Then for any interval $[\delta, \delta'] \subset F$ we have a dichotomy:*

$$C([\delta, \delta'])(t) = C([\delta, \alpha])(t) \frac{1}{(1 - e_{\delta'} t)} \quad (38)$$

or

$$C([\delta, \delta'])(t) = \frac{1}{(1 - e_{\delta'} t)} (e_{\alpha} t C([\delta, \alpha])(t) + C([\delta, \alpha'])(t)) \quad (39)$$

depending on the number of intervals in decomposition (37). The factor $e_{\alpha} t$ corresponds to an interval $[\delta, \alpha]$ in which α is a tail.

For lower bounds we have

$$\begin{aligned} C([\delta, \delta'])(t) &= \frac{1}{(1 - e_{\delta} t)} C([\alpha, \delta])(t) \quad \text{or} \\ C([\delta, \delta'])(t) &= \frac{1}{(1 - e_{\delta} t)} (e_{\alpha} t C([\alpha, \delta])(t) + C([\alpha', \delta])(t)) \end{aligned} \quad (40)$$

Proof. For any $\delta \leq \delta'$ in F we have an identification

$$C_k([\delta, \delta']) = \coprod_{k=l+l'} C_l([\delta, \delta']) \times C_{l'}([\delta', \delta'])$$

Equalities

$$C([\delta, \delta'])(t) = \frac{1}{(1 - e_{\delta'} t)} C([\delta, \delta'])(t) = \frac{1}{(1 - e_{\delta'} t)} C([\delta, \alpha] \cup [\delta, \alpha'])(t) \quad (41)$$

are valid because F is convenient.

F is a lattice therefore

$$C_l([\delta, \alpha] \cup [\delta, \alpha']) = C_l([\delta, \alpha]) \cup C_l([\delta, \alpha']) \quad (42)$$

and

$$C_l([\delta, \alpha]) \cap C_l([\delta, \alpha']) = C_l([\delta, \alpha \wedge \alpha']) \quad (43)$$

If $\alpha = \alpha'$ then equation (41) implies (38). If $\alpha \neq \alpha'$ in (37) then equation (41,42,43) implies

$$C([\delta, \delta'])(t) = \frac{1}{(1 - e_{\delta'} t)} (C([\delta, \alpha])(t) + C([\delta, \alpha'])(t) - C([\delta, \alpha \wedge \alpha'])(t)). \quad (44)$$

Poset F is narrow. Let suppose for certainty α in decomposition (37) is a tail of $[\delta, \alpha]$. Then $[\delta, \alpha] = [\delta, \alpha \wedge \alpha']$ and $C([\delta, \alpha])(t) = \frac{1}{(1 - e_{\alpha} t)} C([\delta, \alpha \wedge \alpha'])(t)$ together with (44) imply (39).

For lower bounds the proof is similar and is omitted. ■

Definition 19 We define an integer-valued function ht on E by the rule: if there is an arrow $\alpha \rightarrow \beta$, then

$$\text{ht}(\alpha) + 1 = \text{ht}(\beta). \quad (45)$$

The function is normalized by condition $\text{ht}((0)) = 0$.

Note that equations (38,39) and an obvious normalization

$$C([\delta, \delta])(t) = \frac{1}{1 - e_\delta t} \quad (46)$$

can be used for inductive computations of $A_\delta^{\delta'}(t)$ because $\text{ht}(\alpha), \text{ht}(\alpha') = \text{ht}(\delta') -$

1. For example $A_{(0)}^{(1)}(t)$ can be computed as follows:

$$\begin{aligned} A_{(0)}^{(1)}(t) &= A_{(0)}^{(2)}(t) \frac{1}{1 - e_{(1)}t} = A_{(0)}^{(3)}(t) \frac{1}{1 - e_{(2)}t} \frac{1}{1 - e_{(1)}t} = \\ &= \left(A_{(0)}^{(45)}(t) A_{(0)}^{(4)}(t) \right) \begin{pmatrix} te_{(45)} \\ 1 \end{pmatrix} \frac{1}{1 - e_{(3)}t} \frac{1}{1 - e_{(2)}t} \frac{1}{1 - e_{(1)}t} = \\ &= \left(A_{(0)}^{(35)}(t) A_{(0)}^{(5)}(t) \right) \begin{pmatrix} \frac{1}{1 - e_{(45)}t} & \frac{1}{1 - e_{(4)}t} \\ 0 & \frac{te_{(5)}}{1 - e_{(4)}t} \end{pmatrix} \begin{pmatrix} te_{(45)} \\ 1 \end{pmatrix} \frac{1}{1 - e_{(3)}t} \frac{1}{1 - e_{(2)}t} \frac{1}{1 - e_{(1)}t} = \\ &= \left(A_{(0)}^{(25)}(t) A_{(0)}^{(34)}(t) \right) \begin{pmatrix} \frac{1}{1 - e_{(35)}t} & 0 \\ \frac{te_{(34)}}{1 - e_{(35)}t} & \frac{1}{1 - e_{(5)}t} \end{pmatrix} \begin{pmatrix} \frac{1}{1 - e_{(45)}t} & \frac{1}{1 - e_{(4)}t} \\ 0 & \frac{te_{(5)}}{1 - e_{(4)}t} \end{pmatrix} \begin{pmatrix} te_{(45)} \\ 1 \end{pmatrix} \frac{1}{1 - e_{(3)}t} \frac{1}{1 - e_{(2)}t} \frac{1}{1 - e_{(1)}t} \\ &\dots \\ &= \frac{1}{1 - e_{(0)}t} \frac{1}{1 - e_{(12)}t} \frac{1}{1 - e_{(13)}t} \begin{pmatrix} \frac{1}{1 - e_{(14)}t} & \frac{1}{1 - e_{(25)}t} \end{pmatrix} \times \\ &\times \begin{pmatrix} \frac{1}{1 - e_{(15)}t} & \frac{1}{1 - e_{(24)}t} \\ 0 & \frac{te_{(23)}}{1 - e_{(24)}t} \end{pmatrix} \begin{pmatrix} \frac{te_{(15)}}{1 - e_{(25)}t} & 0 \\ \frac{1}{1 - e_{(25)}t} & \frac{1}{1 - e_{(34)}t} \end{pmatrix} \times \\ &\times \begin{pmatrix} \frac{1}{1 - e_{(35)}t} & 0 \\ \frac{te_{(34)}}{1 - e_{(35)}t} & \frac{1}{1 - e_{(5)}t} \end{pmatrix} \begin{pmatrix} \frac{1}{1 - e_{(45)}t} & \frac{1}{1 - e_{(4)}t} \\ 0 & \frac{te_{(5)}}{1 - e_{(4)}t} \end{pmatrix} \times \\ &\times \begin{pmatrix} te_{(45)} \\ 1 \end{pmatrix} \frac{1}{1 - e_{(3)}t} \frac{1}{1 - e_{(2)}t} \frac{1}{1 - e_{(1)}t} \end{aligned}$$

The product upon substitution $e_\alpha = 1$ becomes $\frac{1+5t+5t^2+t^3}{(1-t)^{11}}$ (cf. [13],[6]).

6 Algebras of Richardson varieties

We start this section with an accurate description of algebra $A_{\hat{\delta}}^{\hat{\delta}'}$, whose projective spectrum by definition is a semi-infinite Richardson variety. In the end we define the inverse limit of $A_{\hat{\delta}}^{\hat{\delta}'}$.

Definition 20 *The algebra $A_{\hat{\delta}}^{\hat{\delta}'}$ is a quotient of $\mathbb{C}[\lambda^{\hat{\alpha}}]$, $\hat{\alpha} \in [\hat{\delta}, \hat{\delta}'] \subset \hat{E}$. The ideal of relations is generated by the coefficients $\Gamma^{\hat{s}}$ of the generating function (8) where*

$$\lambda^{\alpha}(z) = \sum_{\alpha' \in [\hat{\delta}, \hat{\delta}']} \lambda^{\alpha'} z^l \quad (47)$$

The set of equations (7,8) define an affine variety that parametrizes the curves of degree $\leq N' - N$ is the affine cone (25). We shall refer to equations (8) as the *affinization* of equations (25).

There is a surjective homomorphisms $A_{\hat{\delta}}^{\hat{\delta}'} \rightarrow A_{\hat{\gamma}}^{\hat{\gamma}'}$ $\hat{\delta} \leq \hat{\gamma} \leq \hat{\gamma}' \leq \hat{\delta}'$, whose kernel is generated by $\lambda_{\hat{\alpha}} \in A_{\hat{\delta}}^{\hat{\delta}'}$, $\hat{\alpha} \in [\hat{\delta}, \hat{\delta}'] \setminus [\hat{\gamma}, \hat{\gamma}']$. The homomorphism acts as an identity on the remaining generators. This makes collection $A_{\hat{\delta}}^{\hat{\delta}'}$ an inverse system.

Let \hat{P}' be a polynomial algebra $\mathbb{C}[\lambda_{\hat{\alpha}}]$, $\hat{\alpha} \in \hat{E}$. We define a topology on \hat{E} by declaring a basis of open sets to be $\{T^l[(0), \infty) | l \in \mathbb{Z}\}$ and $\{T^l(-\infty, (0)] | l \in \mathbb{Z}\}$. We denote completion of \hat{P}' with respect to this topology by \hat{P} . It contains a linear space of homogenous elements \hat{P}_k of degree k . An ideal I in \hat{P} topologically generated by expression (8) with no restrictions on l, l' and k . We denote by \hat{A} the quotient \hat{P}/I . There is a homomorphism $\hat{A} \rightarrow A_{\hat{\delta}}^{\hat{\delta}'}$, that acts as an identity on $\lambda_{\hat{\alpha}}$, $\hat{\alpha} \in [\hat{\delta}, \hat{\delta}'] \subset \hat{E}$ and by zero on $\hat{E} \setminus [\hat{\delta}, \hat{\delta}']$

Proposition 21 *There is an isomorphism $e : \hat{A} \rightarrow \varprojlim A_{\hat{\delta}}^{\hat{\delta}'}$*

Proof. We define $P_{\hat{\delta}}^{\hat{\delta}'}$ to be a polynomial algebra $\mathbb{C}[\lambda_{\hat{\alpha}}]$, $\hat{\alpha} \in [\hat{\delta}, \hat{\delta}']$. Then $P_{\hat{\delta}}^{\hat{\delta}'} = \bigoplus_{k \geq 0} P_{\hat{\delta}}^{\hat{\delta}'} k$ is a decomposition into graded components. Let $I_{\hat{\delta}}^{\hat{\delta}'}$ be the kernel of the homomorphism $P_{\hat{\delta}}^{\hat{\delta}'} \rightarrow A_{\hat{\delta}}^{\hat{\delta}'}$. The map $r_{\hat{\delta}, \hat{\delta}', \hat{\gamma}, \hat{\gamma}'} : P_{\hat{\delta}}^{\hat{\delta}'} \rightarrow P_{\hat{\gamma}}^{\hat{\gamma}'}$ is defined on the generators by the same formulas as the map $A_{\hat{\delta}}^{\hat{\delta}'} \rightarrow A_{\hat{\gamma}}^{\hat{\gamma}'}$. The maps $r_{\hat{\delta}, \hat{\delta}', \hat{\gamma}, \hat{\gamma}'}$ are surjective. Moreover the maps $r_{\hat{\delta}, \hat{\delta}', \hat{\gamma}, \hat{\gamma}'} : I_{\hat{\delta}}^{\hat{\delta}'} \rightarrow I_{\hat{\gamma}}^{\hat{\gamma}'}$ are surjective

(this can be checked on generators of degree two). The vanishing of $\varprojlim^1 I_{\hat{\delta}}^{\hat{\delta}'}$ follows (see [24] Section 3.2) from this condition. We have a short exact sequence of limits

$$0 \rightarrow \varprojlim I_{\hat{\delta}}^{\hat{\delta}'} \rightarrow \varprojlim P_{\hat{\delta}}^{\hat{\delta}'} \rightarrow \varprojlim A_{\hat{\delta}}^{\hat{\delta}'} \rightarrow 0$$

For this reason $\hat{P} = \varprojlim P_{\hat{\delta}}^{\hat{\delta}'}$, $I = \varprojlim I_{\hat{\delta}}^{\hat{\delta}'}$ (see [24] about completions). This together with exact sequence implies the isomorphism property of e .

■

7 Properties of \hat{E} as partially ordered set

To proceed with our study of algebras $A_{\hat{\delta}}^{\hat{\delta}'}$ we need to set a terminology and to establish some simple properties of \hat{E} that will be used later. This what we shall do presently.

The set of vertices of the graph \hat{E} (9) is a union

$$\hat{E} = \coprod_{r \in \mathbb{Z}} E_r \quad E_r = \{\alpha^r | \alpha \in E\} \quad (48)$$

E_r define full subgraphs in \hat{E} . We extend function ht (Definition 19) to \hat{E} . It is a matter of simple inspection of (9) to arrive at the following table of values of ht :

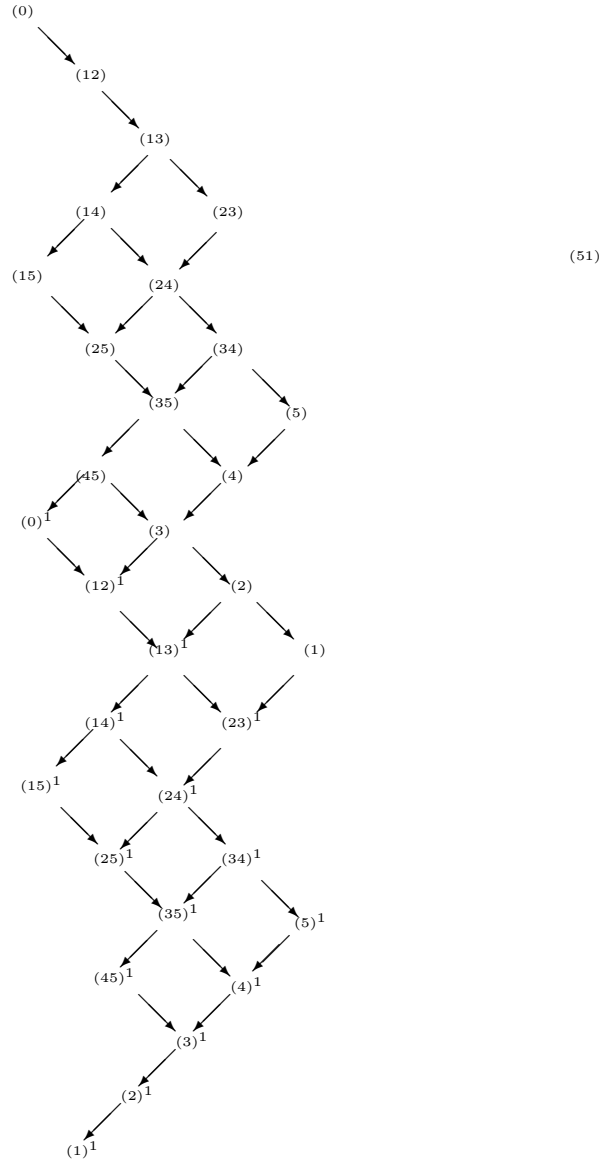
$\hat{\delta}'$	$\hat{\delta}''$	ht
$(0)^r$	$(3)^{r-1}$	$8r$
$(12)^r$	$(2)^{r-1}$	$8r + 1$
$(13)^r$	$(1)^{r-1}$	$8r + 2$
$(14)^r$	$(23)^r$	$8r + 3$
$(15)^r$	$(24)^r$	$8r + 4$
$(25)^r$	$(34)^r$	$8r + 5$
$(35)^r$	$(5)^r$	$8r + 6$
$(45)^r$	$(4)^r$	$8r + 7$

A shift operator T acts on \hat{E} :

$$T\alpha^r = \alpha^{r+1} \quad \alpha^r \in \hat{E} \quad (50)$$

It is an automorphism of the lattice \hat{E} and $T^r E = E_r$.

The poset E coincides with the interval $[(0), (1)] \subset \hat{E}$. Visual presentation of the subinterval $[(0), (1)^1]$ is given below:



Let A, B be two subsets of \hat{E} . We write $A \leqslant B$ if $a \leqslant b$ for all $a \in A, b \in B$.

Lemma 22 *The sets E_r satisfy*

$$E_{r+k} \geq E_r, E_{r-k} \leq E_r \text{ for } k \geq 2$$

Proof. The reader can convince himself by examining the diagram (9). ■

As a corollary we obtain that if $(\hat{\alpha}, \hat{\beta})$ is a clutter then $(\hat{\alpha}, \hat{\beta})$ or $(\hat{\alpha}, T\hat{\beta})$ or $(\hat{\alpha}, T^{-1}\hat{\beta})$ are in E_r .

Proposition 23 *Poset \hat{E} and all its subintervals are lattices.*

Proof. It suffices to verify the statement only for \hat{E} . We need to check that lower bound $\inf[\hat{\alpha}, \infty) \cup [\hat{\beta}, \infty)$ consists of one element; the same should hold for $\sup(-\infty, \hat{\alpha}] \cup (-\infty, \hat{\beta}]$. We know (Lemma 22) that if $\hat{\alpha} \in E_{r+k}$ and $\hat{\beta} \in E_r, |k| \geq 2$ are comparable, $\# \inf[\hat{\alpha}, \infty) \cup [\hat{\beta}, \infty) = 1$ and $\hat{\alpha} \vee \hat{\beta}$ is defined. This argument also works for $\hat{\alpha} \wedge \hat{\beta}$.

By virtue of Lemma 22 and automorphism T we may assume without loss of generality that a pair of non comparable elements $\hat{\alpha}, \hat{\beta}$ belong to $[(0), (1)^1]$, for which lattice property can be verified by direct inspection (see Picture (51)). ■

7.1 Description of the affine Weyl group $W(\hat{D}_5)$

We shall occupy ourself in this section with a description of the affine Weyl group $W(\hat{D}_5)$ as a semidirect product and through generators and relations.

The group $W(\hat{D}_5)$ is a semidirect product $W(D_5) \ltimes X_*$ (cf. [17]), where X_* is a coroot lattice of $SO(10)$. The coroot lattice $X_* SO(10)$ coincides with the kernel of the homomorphism $\mathbb{Z}^5 \rightarrow \mathbb{Z}_2$ defined by the formula

$$\mathbf{m} = [m_1, \dots, m_5] \rightarrow \sum_{i=1}^5 m_i \pmod{2}$$

This homomorphism is nothing else but a homomorphism of fundamental groups $\pi_1(\mathbf{T}^5) \rightarrow \pi_1(SO(10))$ induced by the inclusion of the maximal torus.

Semidirect product $S_5 \ltimes (\mathbb{Z}_2)^5$ acts on \mathbb{Z}^5 . The $(\mathbb{Z}_2)^5$ -factor act coordinate-wise on $\mathbf{m} = [m_1, \dots, m_5] \in \mathbb{Z}^5$ by changing signs. Symmetric group shuffles components of \mathbf{m} .

It is easy to see that the product $s_6 = \sigma_{4,5}(0, 0, 0, 1, 1)[0, 0, 0, 1, 1] \in S_5NX_* = W(\hat{D}_5)$ is an involution. The involutive generators of $W(\hat{D}_5)$ can be arranged into the following Coxeter diagram, which makes $W(\hat{D}_5)$ a Coxeter group:

$$\begin{array}{ccccccc} s_1 & \text{---} & s_3 & \text{---} & s_4 & \text{---} & s_5 \\ & & | & & | & & \\ & & s_2 & & s_6 & & \end{array}$$

We define the action of $W(\hat{D}_5)$ on the set

$$\hat{N} = N \times \mathbb{Z} \quad (52)$$

by the formula:

$$(\sigma, \epsilon, \mathbf{m})(\eta, n) = (\sigma(\epsilon + \eta), -\frac{1}{2} \sum_{i=1}^5 (-1)^{n_i} m_i + n) \quad (53)$$

The group $W(\hat{D}_5)$ acts transitively on \hat{N} .

Proposition 24 *There is an isomorphism of non-directed graphs*

$$Q(\hat{N}, s_1, \dots, s_6) \cong \hat{E}.$$

Proof. If we utilize only s_1, \dots, s_5 we would get a disjoint union (48) (cf. discussion in Section 3). Note that on $(14)^r, (24)^r, (34)^r, (5)^r$ the actions of s_6 and s_5 coincide. We conclude from formula (53) that $s_6(45)^r = (0)^{r+1}, s_6(3)^r = (12)^{r+1}, s_6(2)^r = (13)^{r+1}, s_6(1)^r = (23)^{r+1}$. This verifies the claim. ■

7.2 Interpretation of \hat{E} in terms of the spinor representation $S[z, z^{-1}]$

The Lie-theoretic construction of \hat{E} parallels the one of E . We extend operators R_1^+, \dots, R_5^+ (22,23) to $S[z, z^{-1}]$ by $\mathbb{C}[z, z^{-1}]$ -linearity. Together with

$$R_6^+ = v_4 v_5 z$$

R_1^+, \dots, R_5^+ correspond to simple root vectors in the affine $\hat{\mathfrak{so}}_{10}$.

Proposition 25 *We identify vertices of the graph \hat{E} with weights of representation $S[z, z^{-1}]$. The operators R_0^+, \dots, R_5^+ intertwine the weight spaces. The corresponding Hasse diagram coincides with \hat{E} .*

Proof. By construction, the Hasse diagram of $S[z, z^{-1}]$ is a disjoint union (48) with E_r identified with the diagram of $S \otimes z^r$. It remains to make use of R_6^+ , which is left as an exercise for the reader. ■

In our applications we shall need to know the structure of the set of clutters \hat{K} in \hat{E} . The set \hat{K} is invariant with respect to translations. This observation enables us to translate any clutter into $[(0), (1)^1]$ and carry its analysis on the case-by-case basis. We conclude that \hat{K} besides shifts of clutters from M (20) contains shifts of

$$\begin{aligned} & \{((0)^1, (5)), ((0)^1, (4)), ((0)^1, (3)), ((0)^1, (2)), ((0)^1, (1)), \\ & ((12)^1, (2)), ((12)^1, (1)), ((13)^1, (1)), ((14)^1, (1)), ((15)^1, (1))\} \end{aligned} \quad (54)$$

Lemma 26 *Clutters in \hat{E} belong to one $W(\hat{D}_5)$ orbit in $\text{Sym}^2 \hat{E}$.*

Proof. Let e_i be $[0, \dots, 0, \overset{i}{1}, 0, \dots, 0] \in X_*$. Denote $e_i + e_j, i \neq j$ by \mathbf{m}_{ij} . We can conclude from formula (53) that $\mathbf{m}_{ij}((0)^1, (i)) = ((0), (i)) (-\mathbf{m}_{ij})((ij)^1, (j)) = ((ij), (j))$. We finish the proof as in Proposition 2. ■

8 Proof of the straightened law

This is a principal section of the present paper. Recall that a proof of straightened law should consists of two parts. The easy part establishes the form (2) of relations. The hard part ensures that standard monomials define a basis. We start with the easy part.

Proposition 27 *Each quadric $\Gamma^{\hat{s}}$ in the defining relations of algebras $A_{\hat{\delta}}^{\hat{\delta}'}$ (Definition 20) contains a unique clutter. More over, for any clutter $\lambda^{\hat{\alpha}} \lambda^{\hat{\alpha}'}$ straightened relation (27) holds.*

Proof. We deduce statements of the proposition from the analogous statements for algebra \hat{A} . We follow closely the arguments of Proposition 6.

A choice of complementary pair of components of gamma-maps Γ^+, Γ^- (as in the proof of Proposition 6) and a subordinated choice of \mathfrak{so}_8 , which leaves Γ^+, Γ^- invariant, allows us to construct a set of relations $\Gamma^{+,r}, \Gamma^{-,r}, r \in \mathbb{Z}$. By virtue of (29) we have an \mathfrak{so}_8 -equivariant identification.

$$S[z, z^{-1}] = S^+[z, z^{-1}] \times S^-[z, z^{-1}]. \quad (55)$$

Let

$$p_{\pm} : S[z, z^{-1}] \rightarrow S^{\pm}[z, z^{-1}]$$

be the projections. From results of Proposition 6 $\Gamma^{+,r} = p_+^* \Gamma^{+,r}|_{S^+[z, z^{-1}]}$, $\Gamma^{-,r} = p_-^* \Gamma^{-,r}|_{S^-[z, z^{-1}]}$. The set of quadrics $\Gamma^{+,r}|_{S^+[z, z^{-1}]}$ is an affinization of the quadric $\Gamma^+|_{S^+}$. This situation has been studied in ([26]). S^+ is equivalent by triality to the fundamental representation V^8 of \mathfrak{so}_8 . As in Proposition 6, we conclude that Hasse diagrams of $V^8[z, z^{-1}]$ and $S^+[z, z^{-1}]$ coincide. We use the fact that the lowest root in the adjoint representation of \mathfrak{so}_8 is invariant with respect to our triality automorphism. This immediately follows from the cross shape of the extended Dynkin diagram corresponding to $\hat{\mathfrak{so}}_8$. The Hasse diagrams \hat{G}_4 of $V^8[z, z^{-1}]$

$$\begin{array}{ccccccc} & & \nearrow (1)^r & \searrow & & \nearrow (4)^{r+1} & \searrow \\ \cdots \rightarrow (3)^r & \rightarrow & (2)^r & & (2^*)^r \rightarrow (3^*)^r & & (3)^{r+1} \rightarrow (2)^{r+1} \rightarrow \cdots \\ & & \searrow (1^*)^r & \nearrow & & \searrow (4^*)^r & \nearrow \end{array}$$

has been determined in [26].

By the results of [26] each equation of affinization of a nondegenerate quadric contains precisely one clutter and relation (27) holds. Arguing as in the proof of Proposition 6 and using result of Lemma 26 we conclude that indices of the clutter remains to be incomparable in \hat{E} . The poset associated with \hat{G}_4 is subposet in \hat{E} . Thus $\Gamma^{+,r}|_{S^+[z, z^{-1}]}$ lead to straightened relation understood in the sense of \hat{E} order.

Let q be the projection $\hat{P} \rightarrow P_{\hat{\delta}}^{\hat{\delta}'}$. Suppose $\lambda^{\hat{\gamma}} \lambda^{\hat{\gamma}'}$ is a monomial in $\Gamma^{\mathbf{s}}$ such that $q(\lambda^{\hat{\gamma}} \lambda^{\hat{\gamma}'}) \neq 0$. Then $\hat{\gamma}, \hat{\gamma}' \in [\hat{\delta}, \hat{\delta}']$. By the above results, if $\hat{\gamma}, \hat{\gamma}'$ is not a clutter itself, the indices of the clutter $\lambda^{\hat{\alpha}} \lambda^{\hat{\alpha}'}$ in $\Gamma^{\mathbf{s}}$ satisfy $\hat{\gamma} \leq \hat{\alpha}, \hat{\alpha}' \leq \hat{\gamma}'$, which implies that $\hat{\alpha}, \hat{\alpha}' \in [\hat{\delta}, \hat{\delta}']$ and $q(\lambda^{\hat{\alpha}} \lambda^{\hat{\alpha}'}) \neq 0$. We conclude that $q(\Gamma^{\mathbf{s}})$ contains a unique clutter. The claim follows because all relations in $A_{\hat{\delta}}^{\hat{\delta}'}$ have this form.

■

The difficult part in the proof of straitened law claims that standard monomials are linearly independent. To prove this, we write relations (8) more explicitly:

$$\begin{aligned}
\Gamma(1)^{2n} &= \underbrace{w_{2,5}^n w_{3,4}^n}_{\square} - \overbrace{w_{2,4}^n w_{3,5}^n}^{\blacksquare} + \dots \sim 0 & \Gamma(2)^{2n} &= -\underbrace{w_{1,5}^n w_{3,4}^n}_{\square} + \overbrace{w_{1,4}^n w_{3,5}^n}^{\blacksquare} + \dots \sim 0 \\
\Gamma(3)^{2n} &= \underbrace{w_{1,5}^n w_{2,4}^n}_{\square} - \overbrace{w_{1,4}^n w_{2,5}^n}^{\blacksquare} + \dots \sim 0 & \Gamma(4)^{2n} &= -\underbrace{w_{1,5}^n w_{2,3}^n}_{\square} + \overbrace{w_{1,3}^n w_{2,5}^n}^{\blacksquare} + \dots \sim 0 \\
\Gamma(5)^{2n} &= \underbrace{w_{1,4}^n w_{2,3}^n}_{\square} - \overbrace{w_{1,3}^n w_{2,4}^n}^{\blacksquare} + \dots \sim 0 & \Gamma(1^*)^{2n} &= -\underbrace{p_4^n w_{1,4}^n}_{\square} + \overbrace{p_5^n w_{1,5}^n}^{\blacksquare} + \dots \sim 0 \\
\Gamma(2^*)^{2n} &= -\underbrace{p_4^n w_{2,4}^n}_{\square} + \overbrace{p_5^n w_{2,5}^n}^{\blacksquare} + \dots \sim 0 & \Gamma(3^*)^{2n} &= -\underbrace{p_4^n w_{3,4}^n}_{\square} + \overbrace{p_5^n w_{3,5}^n}^{\blacksquare} + \dots \sim 0 \\
\Gamma(4^*)^{2n} &= -\underbrace{p_3^n w_{3,4}^n}_{\square} + \overbrace{p_5^n w_{4,5}^n}^{\blacksquare} + \dots \sim 0 & \Gamma(5^*)^{2n} &= -\underbrace{p_3^n w_{3,5}^n}_{\square} + \overbrace{p_4^n w_{4,5}^n}^{\blacksquare} + \dots \sim 0 \\
\Gamma(1)^{2n+1} &= \underbrace{\lambda^{n+1} p_1^n}_{\square} + \overbrace{w_{2,3}^{n+1} w_{4,5}^n}^{\blacksquare} + \dots \sim 0 & \Gamma(2)^{2n+1} &= -\underbrace{\lambda^{n+1} p_2^n}_{\square} - \overbrace{w_{1,3}^{n+1} w_{4,5}^n}^{\blacksquare} + \dots \sim 0 \\
\Gamma(3)^{2n+1} &= \underbrace{\lambda^{n+1} p_3^n}_{\square} + \overbrace{w_{1,2}^{n+1} w_{4,5}^n}^{\blacksquare} + \dots \sim 0 & \Gamma(4)^{2n+1} &= -\underbrace{\lambda^{n+1} p_4^n}_{\square} - \overbrace{w_{1,2}^{n+1} w_{3,5}^n}^{\blacksquare} + \dots \sim 0 \\
\Gamma(5)^{2n+1} &= \underbrace{\lambda^{n+1} p_5^n}_{\square} + \overbrace{w_{1,2}^{n+1} w_{3,4}^n}^{\blacksquare} + \dots \sim 0 & \Gamma(1^*)^{2n+1} &= -\underbrace{p_2^n w_{1,2}^{n+1}}_{\square} + \overbrace{p_3^n w_{1,3}^{n+1}}^{\blacksquare} + \dots \sim 0 \\
\Gamma(2^*)^{2n+1} &= -\underbrace{p_1^n w_{1,2}^{n+1}}_{\square} + \overbrace{p_3^n w_{2,3}^{n+1}}^{\blacksquare} + \dots \sim 0 & \Gamma(3^*)^{2n+1} &= -\underbrace{p_1^n w_{1,3}^{n+1}}_{\square} + \overbrace{p_2^n w_{2,3}^{n+1}}^{\blacksquare} + \dots \sim 0 \\
\Gamma(4^*)^{2n+1} &= -\underbrace{p_1^n w_{1,4}^{n+1}}_{\square} + \overbrace{p_2^n w_{2,4}^{n+1}}^{\blacksquare} + \dots \sim 0 & \Gamma(5^*)^{2n+1} &= -\underbrace{p_1^n w_{1,5}^{n+1}}_{\square} + \overbrace{p_2^n w_{2,5}^{n+1}}^{\blacksquare} + \dots \sim 0
\end{aligned} \tag{56}$$

Monomials marked by underbracket \square are formed by incomparable variables. Monomials $\lambda^{\alpha \wedge \beta} \lambda^{\alpha \vee \beta}$ are marked by overbracket \blacksquare .

Construction a basis $B_{\hat{\delta} \ k}^{\hat{\delta}'}$ in k -th graded component of $A_{\hat{\delta}}^{\hat{\delta}'}$ follows the line of a similar construction for algebra A . Monomials $\hat{\lambda}^{\mathbf{n}} = \prod_{\hat{\alpha} \in [\hat{\delta}, \hat{\delta}']} (\lambda^{\hat{\alpha}})^{n_{\hat{\alpha}}}$, $\deg \lambda^{\mathbf{n}} = \overline{\mathbf{n}} = k$ form a basis in $P_{\hat{\delta} \ k}^{\hat{\delta}'} \subset P_{\hat{\delta}}^{\hat{\delta}'}$. We construct $B_{\hat{\delta} \ k}^{\hat{\delta}'} \subset \{\hat{\lambda}^{\mathbf{n}} | \overline{\mathbf{n}} = k\}$ and then project into $A_{\hat{\delta} \ k}^{\hat{\delta}'}$. We do it in several steps. The first approximation to $B_{\hat{\delta} \ k}^{\hat{\delta}'}$ is a subset $X_{\hat{\delta} \ k}^{\hat{\delta}'} \subset \{\hat{\lambda}^{\mathbf{n}} | \overline{\mathbf{n}} = k\}$ of standard monomials $\hat{\lambda}^{\mathbf{n}}$. Comparable indices define a path in $[\hat{\delta}, \hat{\delta}']$, with weights $n_{\hat{\alpha}}, \sum n_{\hat{\alpha}} = k$ at the vertices. The general theory tells us that there could be sequence of sets

$$X_{\hat{\delta} \ k}^{\hat{\delta}'} \supseteq Y_{\hat{\delta} \ k}^{\hat{\delta}'} \supseteq \dots \supseteq B_{\hat{\delta} \ k}^{\hat{\delta}'}$$

that converges to a basis. We shall prove however that $X_{\hat{\delta} \ k}^{\hat{\delta}' \ } = B_{\hat{\delta} \ k}^{\hat{\delta}' \ }$. We define $\hat{X}_k = X_{-\infty \ k}^\infty$.

Let $v_{\mathbf{s}^r} = v_{\mathbf{s}} z^r, \mathbf{s}^r \in \hat{\mathbf{G}}$ be a basis $V[z, z^{-1}]$ build upon basis (5), and let $(x_{(\mathbf{s})^l}) = (x_{(1)^l}, \dots, x_{(5)^l}, x_{(1^*)^l}, \dots, x_{(5^*)^l}) \quad l \in \mathbb{Z}$ be the dual basis. We note that Fierz identity (33) implies

$$I_\alpha(z) = \Gamma_{\alpha\beta}^i \lambda^\beta(z) \Gamma_{\gamma\delta}^i \lambda^\gamma(z) \lambda^\delta(z) = 0 \quad (57)$$

We define multiplication map by the formula

$$\lambda^{\alpha^l} x_{\mathbf{s}^r} \rightarrow \lambda^{\alpha^l} \Gamma^{\mathbf{s}^r} \quad (58)$$

Expressions

$$h_{\alpha^k} = \sum_{l+l'=k} \sum_{\beta \in \mathbf{E}} \sum_{i=1}^5 \Gamma_{\alpha\beta}^i \lambda^{\beta^{l'}} x_{(i^*)^l} + \Gamma_{\alpha\beta}^{i*} \lambda^{\beta^{l'}} x_{(i)^l} \quad (59)$$

upon substitution (58) by virtue (57) become zero.

The action of operator $T : h_{\alpha^k} \rightarrow h_{\alpha^{k+3}}$ (74) is compatible with its action $T : \Gamma^{\mathbf{s}^l} \rightarrow \Gamma^{\mathbf{s}^{l+2}}$ on quadrics (8) and on generators λ^{α^l} . The operator \hat{u} (73) transforms Fierz identities as follows:

$$\hat{u} : h_{\alpha^k} \rightarrow \pm h_{s(\alpha)-k}.$$

In light of these remarks the following lemma is obvious.

Lemma 28 *Under the group generated by T, \hat{u} any of the expressions (59) can be transformed to h_{α^0} or $h_{\alpha^1}, \alpha \in \mathbf{E}$.*

Proposition 27 characterizes quadrics $\Gamma^{\hat{\mathbf{s}}} \in \hat{P}$ that map to zero in $P_{\hat{\delta}}^{\hat{\delta}'}$ as those that contains clutters with indices not in $[\hat{\delta}, \hat{\delta}']$. Recall that \hat{P} and its graded components are spaces with topology. By $\hat{\otimes}$ we understand the tensor product completed in the topology.

Monomials $\lambda^{\hat{\gamma}} \otimes \lambda^{\hat{\alpha}} \lambda^{\hat{\beta}}$ with unconstrained indices form a topological basis in $\hat{P}_1 \hat{\otimes} \hat{P}_2$.

The map

$$x_{\hat{\mathbf{s}}} \rightarrow \Gamma^{\hat{\mathbf{s}}} \quad (60)$$

identifies the second graded component \hat{I}_2 of the ideal \hat{I} with the space dual to $V[z, z^{-1}]$. This allows us to think of $h_{\hat{\alpha}}$ as elements in $\hat{P}_1 \hat{\otimes} \hat{I}_2 \subset \hat{P} \hat{\otimes} \hat{I}$.

Proposition 29 *An element $h_{\alpha^n} \in \hat{P} \hat{\otimes} \hat{I}$ contains precisely a pair of noncommutative obstructions $\lambda^{\hat{\gamma}} \otimes \lambda^{\hat{\alpha}} \lambda^{\hat{\beta}}, \lambda^{\hat{\gamma}'} \otimes \lambda^{\hat{\alpha}'} \lambda^{\hat{\beta}'}$ (see Definition 8), such that their images coincide in \hat{P} . Conversely, for any pair of noncommutative obstructions $\lambda^{\hat{\gamma}} \otimes \lambda^{\hat{\alpha}} \lambda^{\hat{\beta}} \neq \lambda^{\hat{\gamma}'} \otimes \lambda^{\hat{\alpha}'} \lambda^{\hat{\beta}'}$ with $\lambda^{\hat{\gamma}} \lambda^{\hat{\alpha}} \lambda^{\hat{\beta}} = \lambda^{\hat{\gamma}'} \lambda^{\hat{\alpha}'} \lambda^{\hat{\beta}'}$ there is $h_{\alpha^n} \in \hat{P} \hat{\otimes} \hat{I}$ that contains these obstructions as monomials.*

Proof. The transformations T, \hat{u} act simultaneously on the ideal \hat{I} and on the poset \hat{E} and induce transformations on the sets of clutters and obstructions. By the results of Lemma 28, it suffices to analyze h_{α^0} and h_{α^1} .

The content of h_{α^1} is tabulated below.

$$\begin{aligned}
h_{(0)1} &= p_1^1 x_{(1*)0} - p_2^1 x_{(2*)0} + p_3^1 x_{(3*)0} - p_4^1 x_{(4*)0} + p_5^1 x_{(5*)0} + \underline{p_1^0 x_{(1*)1} - p_2^0 x_{(2*)1} + p_3^0 x_{(3*)1} - p_4^0 x_{(4*)1} + p_5^0 x_{(5*)1}} \cdots = 0 \\
h_{(12)1} &= w_{4,5}^1 x_{(3*)0} - w_{3,5}^1 x_{(4*)0} + w_{3,4}^1 x_{(5*)0} - p_2^1 x_{(1)0} - p_1^1 x_{(2)0} + w_{4,5}^0 x_{(3*)1} - w_{3,5}^0 x_{(4*)1} + w_{3,4}^0 x_{(5*)1} - \underline{p_2^0 x_{(1)1} - p_1^0 x_{(2)1}} \cdots = 0 \\
h_{(13)1} &= -w_{4,5}^1 x_{(2*)0} + w_{2,5}^1 x_{(4*)0} - w_{2,4}^1 x_{(5*)0} + p_3^1 x_{(1)0} - p_1^1 x_{(3)0} - w_{4,5}^0 x_{(2*)1} + w_{2,5}^0 x_{(4*)1} - w_{2,4}^0 x_{(5*)1} + \underline{p_3^0 x_{(1)1} - p_1^0 x_{(3)1}} \cdots = 0 \\
h_{(23)1} &= w_{4,5}^1 x_{(1*)0} - w_{1,5}^1 x_{(4*)0} + w_{1,4}^1 x_{(5*)0} + p_3^1 x_{(2)0} + p_2^1 x_{(3)0} + w_{4,5}^0 x_{(1*)1} - w_{1,5}^0 x_{(4*)1} + w_{1,4}^0 x_{(5*)1} + \underline{p_3^0 x_{(2)1} + p_2^0 x_{(3)1}} \cdots = 0 \\
h_{(14)1} &= w_{3,5}^1 x_{(2*)0} - w_{2,5}^1 x_{(3*)0} + w_{2,3}^1 x_{(5*)0} - p_4^1 x_{(1)0} - p_1^1 x_{(4)0} + w_{3,5}^0 x_{(2*)1} - w_{2,5}^0 x_{(3*)1} + w_{2,3}^0 x_{(5*)1} - \underline{p_4^0 x_{(1)1} - p_1^0 x_{(4)1}} \cdots = 0 \\
h_{(24)1} &= -w_{3,5}^1 x_{(1*)0} + w_{1,5}^1 x_{(3*)0} - w_{1,3}^1 x_{(5*)0} - p_4^1 x_{(2)0} + p_2^1 x_{(4)0} - w_{3,5}^0 x_{(1*)1} + w_{1,5}^0 x_{(3*)1} - w_{1,3}^0 x_{(5*)1} - \underline{p_4^0 x_{(2)1} + p_2^0 x_{(4)1}} \cdots = 0 \\
h_{(15)1} &= -w_{3,4}^1 x_{(2*)0} + w_{2,4}^1 x_{(3*)0} - w_{2,3}^1 x_{(4*)0} + p_5^1 x_{(1)0} - p_1^1 x_{(5)0} - w_{3,4}^0 x_{(2*)1} + w_{2,4}^0 x_{(3*)1} - w_{2,3}^0 x_{(4*)1} + \underline{p_5^0 x_{(1)1} - p_1^0 x_{(5)1}} \cdots = 0 \\
h_{(34)1} &= w_{2,5}^1 x_{(1*)0} - w_{1,5}^1 x_{(2*)0} + w_{1,2}^1 x_{(5*)0} - p_4^1 x_{(3)0} - p_3^1 x_{(4)0} + w_{2,5}^0 x_{(1*)1} - w_{1,5}^0 x_{(2*)1} + w_{1,2}^0 x_{(5*)1} - \underline{p_4^0 x_{(3)1} - p_3^0 x_{(4)1}} \cdots = 0 \\
h_{(25)1} &= w_{3,4}^1 x_{(1*)0} - w_{1,4}^1 x_{(3*)0} + w_{1,3}^1 x_{(4*)0} + p_5^1 x_{(2)0} + p_2^1 x_{(5)0} + w_{3,4}^0 x_{(1*)1} - w_{1,4}^0 x_{(3*)1} + w_{1,3}^0 x_{(4*)1} + \underline{p_5^0 x_{(2)1} + p_2^0 x_{(5)1}} \cdots = 0 \\
h_{(5)1} &= \lambda^1 x_{(5*)0} + w_{1,5}^1 x_{(1)0} + w_{2,5}^1 x_{(2)0} + w_{3,5}^1 x_{(3)0} + w_{4,5}^1 x_{(4)0} + \lambda^0 x_{(5*)1} + w_{1,5}^0 x_{(1)1} + w_{2,5}^0 x_{(2)1} + w_{3,5}^0 x_{(3)1} + \underline{w_{4,5}^0 x_{(4)1}} \cdots = 0 \\
h_{(35)1} &= -w_{2,4}^1 x_{(1*)0} + w_{1,4}^1 x_{(2*)0} - w_{1,2}^1 x_{(4*)0} + p_5^1 x_{(3)0} - p_3^1 x_{(5)0} - w_{2,4}^0 x_{(1*)1} + w_{1,4}^0 x_{(2*)1} - w_{1,2}^0 x_{(4*)1} + \underline{p_5^0 x_{(3)1} - p_3^0 x_{(5)1}} \cdots = 0 \\
h_{(4)1} &= -\lambda^1 x_{(4*)0} - w_{1,4}^1 x_{(1)0} - w_{2,4}^1 x_{(2)0} - w_{3,4}^1 x_{(3)0} + w_{4,5}^1 x_{(5)0} - \lambda^0 x_{(4*)1} - w_{1,4}^0 x_{(1)1} - w_{2,4}^0 x_{(2)1} - w_{3,4}^0 x_{(3)1} + \underline{w_{4,5}^0 x_{(5)1}} \cdots = 0 \\
h_{(45)1} &= w_{2,3}^1 x_{(1*)0} - w_{1,3}^1 x_{(2*)0} + w_{1,2}^1 x_{(3*)0} + p_5^1 x_{(4)0} + p_4^1 x_{(5)0} + w_{2,3}^0 x_{(1*)1} - w_{1,3}^0 x_{(2*)1} + w_{1,2}^0 x_{(3*)1} + \underline{p_5^0 x_{(4)1} + p_4^0 x_{(5)1}} \cdots = 0 \\
h_{(3)1} &= \lambda^1 x_{(3*)0} + w_{1,3}^1 x_{(1)0} + w_{2,3}^1 x_{(2)0} - w_{3,4}^1 x_{(4)0} - w_{3,5}^1 x_{(5)0} + \lambda^0 x_{(3*)1} + w_{1,3}^0 x_{(1)1} + w_{2,3}^0 x_{(2)1} - w_{3,4}^0 x_{(4)1} - \underline{w_{3,5}^0 x_{(5)1}} \cdots = 0 \\
h_{(2)1} &= -\lambda^1 x_{(2*)0} - w_{1,2}^1 x_{(1)0} + w_{2,3}^1 x_{(3)0} + w_{2,4}^1 x_{(4)0} + w_{2,5}^1 x_{(5)0} - \lambda^0 x_{(2*)1} - w_{1,2}^0 x_{(1)1} + w_{2,3}^0 x_{(3)1} + w_{2,4}^0 x_{(4)1} + \underline{w_{2,5}^0 x_{(5)1}} \cdots = 0 \\
h_{(1)1} &= \lambda^1 x_{(1*)0} - w_{1,2}^1 x_{(2)0} - w_{1,3}^1 x_{(3)0} - w_{1,4}^1 x_{(4)0} - w_{1,5}^1 x_{(5)0} + \lambda^0 x_{(1*)1} - w_{1,2}^0 x_{(2)1} - w_{1,3}^0 x_{(3)1} - w_{1,4}^0 x_{(4)1} - \underline{w_{1,5}^0 x_{(5)1}} \cdots = 0
\end{aligned}$$

(61)

The underlined terms after substitution (60) contain complementary pairs of noncommutative obstructions (76). The reader can check using equations (8) that noncommutative obstructions have coefficients equal to ± 1 and have oppo-

site signs. There is another way to become convinced in the choice of coefficients of obstructing monomials: the image of $h_{\hat{\alpha}}$ in \hat{P} is zero and for this coefficients of obstructions must have opposite signs. The omitted terms in (61) do not contribute to the set of obstructions because of results of Lemma 22 and explicit description of clutters in $\Gamma^{\hat{s}}$ (8)

Zero modes in h_{α^0} coincide with h_{α} (35). Analysis of the remaining modes in h_{α^0} can be carried out along the lines of h_{α^1} case.

The converse statement of the proposition can be proved as follows. First, take an obstruction and transform it to an obstruction in $[(0), (1)^1]$ by means of T and \hat{u} . Then, a case-by-case study (Appendix B) reveals that obstruction in $[(0), (1)^1]$ are in one-to-one correspondence to a noncommutative obstructions in some h_{α^0} , h_{α^1} or h_{α^2} . The h_{α^2} -case as we already know can be reduced to h_{α^1} -case by means of automorphisms T and \hat{u} . ■

Proposition 30 *For any pair of complementary noncommutative obstruction $\lambda^{\hat{\alpha}} \otimes \lambda^{\hat{\beta}} \lambda^{\hat{\gamma}}, \lambda^{\hat{\alpha}'} \otimes \lambda^{\hat{\beta}'} \lambda^{\hat{\gamma}'} \in P_{\hat{\delta}}^{\hat{\delta}'} \otimes I_{\hat{\delta}}^{\hat{\delta}'}$ there is $h_{\hat{\alpha}}$ (59) such that after the substitution (60) and projection $p : P \rightarrow P_{\hat{\delta}}^{\hat{\delta}'}$ expression $h_{\hat{\alpha}}$ transforms to an expression that contains monomials $\lambda^{\hat{\alpha}} \otimes \lambda^{\hat{\beta}} \lambda^{\hat{\gamma}}, \lambda^{\hat{\alpha}'} \otimes \lambda^{\hat{\beta}'} \lambda^{\hat{\gamma}'}$.*

Proof. The map p acts as an identity on $\lambda^{\hat{\alpha}}, \hat{\alpha} \in [\hat{\delta}, \hat{\delta}']$ and by zero on $\lambda^{\hat{\alpha}}, \hat{\alpha} \notin [\hat{\delta}, \hat{\delta}']$. Only monomials $\prod_i \lambda^{\hat{\alpha}_i}$ with $\hat{\alpha}_i \in [\hat{\delta}, \hat{\delta}']$ survive under the map p . A noncommutative obstruction defines two $[\hat{\delta}, \hat{\delta}']$ -clutters. They are still clutters in \hat{E} . Hence $\lambda^{\hat{\alpha}} \otimes \lambda^{\hat{\beta}} \lambda^{\hat{\gamma}}, \lambda^{\hat{\alpha}'} \otimes \lambda^{\hat{\beta}'} \lambda^{\hat{\gamma}'}$ is a complementary pair of noncommutative obstructions in $\hat{P} \hat{\otimes} \hat{I}$. Hence by Proposition 29 there is a $h_{\hat{\alpha}} \in \hat{P} \hat{\otimes} \hat{I}$. Its image in $P_{\hat{\delta}}^{\hat{\delta}'} \otimes I_{\hat{\delta}}^{\hat{\delta}'}$ contains $\lambda^{\hat{\alpha}} \otimes \lambda^{\hat{\beta}} \lambda^{\hat{\gamma}}$. ■

We define a strict total order on the set of generators of \hat{P} , which refines a partial order defined on \hat{E} . The operator T preserves the order:

$$\begin{aligned} \dots < (4)^{r-1} < (0)^r < (3)^{r-1} < (12)^r < (2)^{r-1} < (13)^r < (1)^{r-1} < (14)^r < (23)^r < \\ < (15)^r < (24)^r < (25)^r < (34)^r < (35)^r < (5)^r < (45)^r < (4)^r < (0)^{r+1} < \dots \end{aligned} \quad (62)$$

The total order on the set of generators of $P_{\hat{\delta}}^{\hat{\delta}'}$ is the restriction of the above. The set of relations (31) is compatible with (62).

Proposition 31 *If for any pair $\Gamma^{\hat{s}}, \Gamma^{\hat{s}'} \in I_{\hat{\delta}}^{\hat{\delta}'} \subset P_{\hat{\delta}}^{\hat{\delta}'}$ (8) with $T(\Gamma^{\hat{s}}, \Gamma^{\hat{s}'}) \neq T(\Gamma^{\hat{s}})T(\Gamma^{\hat{s}'})$ the S -polynomials are*

$$S(\Gamma^{\hat{s}}, \Gamma^{\hat{s}'}) = \sum_{\hat{\alpha} \in \hat{E}} \sum_{\hat{r} \in \hat{G}} c_{\hat{\alpha}, \hat{r}} \lambda^{\hat{\alpha}} \Gamma^{\hat{r}} \quad (63)$$

with no obstructions among monomials in $\lambda^{\hat{\alpha}} \Gamma^{\hat{r}}$, then the set $X_{\hat{\delta}}^{\hat{\delta}'}_k$ defines a basis in $A_{\hat{\delta}}^{\hat{\delta}'}_k$.

Proof. The proof is based on the result of Proposition 30. We omit it because it is similar to the proof of Proposition 12. ■

Corollary 32 *The algebra $A_{\hat{\delta}}^{\hat{\delta}'}$ is an algebra with straightened law.*

Proof. The same as of Proposition 12, though based on Propositions 27 and 31. ■

Corollary 33 *Let us fix the order (31) on generators of $P_{\hat{\delta}}^{\hat{\delta}'}$. Then the relations (8) form a quadratic Gröbner basis in the ideal $I_{\hat{\delta}}^{\hat{\delta}'}$. In particular algebras $A_{\hat{\delta}}^{\hat{\delta}'}$ are Koszul.*

Proof. By virtue of general theory [25] [27] result follows from (31). ■

Proposition 34 1. *$\text{Spec}(A_{\hat{\delta}}^{\hat{\delta}'})$ is a reduced affine scheme.*

$$2. \dim \text{Spec}(A_{\hat{\delta}}^{\hat{\delta}'}) = \text{ht}(\hat{\delta}') - \text{ht}(\hat{\delta}).$$

Proof. The proof follows from Corollary 3.6 in [14]. Corollary reduces computation of dimension to finding the length of the maximal chain $\alpha_1 < \dots < \alpha_d$ in $[\hat{\delta}, \hat{\delta}']$. For $[\hat{\delta}, \hat{\delta}'] \subset \hat{E}$ it is expressible in terms of ht (45,49) and coincides with $\text{ht}(\hat{\delta}') - \text{ht}(\hat{\delta})$. ■

Depth $\text{depth}(A_{\hat{\delta}}^{\hat{\delta}'})$ is defined as a length of the maximal regular sequence in the ideal in $A_{\hat{\delta}}^{\hat{\delta}'}$ generated by $\lambda^{\hat{\alpha}}$.

Proposition 35

$$\text{depth}(A_{\hat{\delta}}^{\hat{\delta}'}) = \dim \text{Spec}(A_{\hat{\delta}}^{\hat{\delta}'}) = \text{ht}(\hat{\delta}') - \text{ht}(\hat{\delta})$$

We set

$$y_i = \sum_{\text{ht}(\hat{\alpha})=i} \lambda^{\hat{\alpha}},$$

with ht defined in (45). Then $y_{\text{ht}(\hat{\delta})}, \dots, y_{\text{ht}(\hat{\delta}')}$ form a regular sequence in $A_{\hat{\delta}}^{\hat{\delta}'}$.

Proof. An element β in F is a *cover* of α in F if $\beta > \alpha$ and no elements lies strictly in between α and β .

The poset F is called *wonderful* [14] if in the poset $H \cup \{\infty\} \cup \{-\infty\}$ obtained by adjoining greatest and the least elements (if they are not already present), the following condition holds: if $\beta_1, \beta_2 < \gamma$ are covers of an element α , then there is an element $\beta < \gamma$, which covers both β_1 and β_2 .

An easy inspection shows that \hat{E} and $[\hat{\delta}, \hat{\delta}']$ are wonderful. The proposition follows Theorem 4.1 in [14], which establish this in a greater generality for algebras with straightened law over a wonderful poset. ■

Corollary 36 $A_{\hat{\delta}}^{\hat{\delta}'}$ is a Cohen-Macaulay algebra.

Proof. Follows from Corollary 4.2 [14]. ■

9 Computation of Poincaré series $A_{\hat{\delta}}^{\hat{\delta}'}(z, q, t)$

Chain counting method that we developed for the purpose of computation of character of pure spinors is fairly general and relies on combinatorics of Hasse diagram. This is why we can carry over most of the results of Section 5 to setting of algebras $A_{\hat{\delta}}^{\hat{\delta}'}$.

To be precise by the result of Corollary 32 standard monomials form a basis in $A_{\hat{\delta}}^{\hat{\delta}'}$. Standard monomials are in one-to-one correspondence with chains in $[\hat{\delta}, \hat{\delta}'] \subset \hat{E}$. A generating function $A_{\hat{\delta}}^{\hat{\delta}'}(z, q, t)$ that encodes Lie-algebraic information about $A_{\hat{\delta}}^{\hat{\delta}'}$ is a specialization of $C([\hat{\delta}, \hat{\delta}'])(t)$ (36).

To be precise, graded components of $A_{\hat{\delta}}^{\hat{\delta}'}$ are representations of a maximal torus $H \subset \text{Spin}(10)$; \mathbb{C}^\times acts on coefficients of $\lambda^\alpha(z)$ (47) via reparametrization:

$$\lambda^\alpha(z) \rightarrow \lambda^\alpha(qz).$$

Let $a_n(z, q)$ be the character of the $H \times \mathbb{C}^\times$ represented in n -th graded component of $A_{\hat{\delta}}^{\hat{\delta}'}$. We define

$$A_{\hat{\delta}}^{\hat{\delta}'}(z, q, t) = \sum_{n \geq 0} a_n(z, q) t^n.$$

We conclude, as in Section 5, that $C([\hat{\delta}, \hat{\delta}'])(t)$, upon substitution

$$e_{\alpha^r} = q^r e_\alpha(z_1, \dots, z_5), \quad (64)$$

becomes $A_{\hat{\delta}}^{\hat{\delta}'}(t, z, q)$.

The isomorphism $T^N : A_{\hat{\delta}}^{\hat{\delta}'} \rightarrow A_{T^N \hat{\delta}}^{T^N \hat{\delta}'}$ shifts \mathbb{C}^\times -weights of all generators on N and

$$A_{\alpha^N}^{\beta^{N'}}(z, q, t) = A_\alpha^{\beta^{N'-N}}(z, q, q^N t) \quad \alpha, \beta \in E \quad (65)$$

It is convenient to group generating functions in arrays

$$(C([\hat{\delta}, \hat{\delta}'])(t) \ C([\hat{\delta}, \hat{\delta}'])(t)) \quad \hat{\delta}' \neq \hat{\delta}'' \quad \text{ht}(\hat{\delta}') = \text{ht}(\hat{\delta}'') = l$$

(cf. 49). The ordered pair $(\hat{\delta}', \hat{\delta}'')$ shall be called a *complementary* pair of weights. Lattice \hat{E} is convenient and narrow (see Section 5). The formulas (38 39) can be written in a matrix form

$$\begin{aligned} (C([\hat{\delta}, \hat{\delta}'])(t) \ C([\hat{\delta}, \hat{\delta}'])(t)) &= (C([\hat{\delta}, \hat{\epsilon}'])(t) \ C([\hat{\delta}, \hat{\epsilon}''])(t)) U_l \\ \text{ht}(\hat{\epsilon}') + 1 = \text{ht}(\hat{\epsilon}'') + 1 &= \text{ht}(\hat{\delta}') = \text{ht}(\hat{\delta}'') = l \end{aligned} \quad (66)$$

Upon substitution (64), $U_l(z, q, t)$ satisfies

$$U_{l+8}(z, q, t) = U_l(z, q, qt)$$

This enables to effectively recover all $U_l(z, q, t)$ from the first eight matrices:

$$\begin{aligned}
U_1(z, q, t) &= \begin{pmatrix} \frac{e_{(0)} t}{1-e_{(12)} t} & 0 \\ \frac{1}{1-e_{(12)} t} & \frac{1}{1-e_{(2)} q^{-1} t} \end{pmatrix} & U_2(z, q, t) &= \begin{pmatrix} \frac{1}{1-e_{(13)} t} & 0 \\ \frac{e_{(2)} q^{-1} t}{1-e_{(13)} t} & \frac{1}{1-e_{(1)} q^{-1} t} \end{pmatrix} \\
U_3(z, q, t) &= \begin{pmatrix} \frac{1}{1-e_{(14)} t} & \frac{1}{1-e_{(23)} t} \\ 0 & \frac{e_{(1)} q^{-1} t}{1-e_{(23)} t} \end{pmatrix} & U_4(z, q, t) &= \begin{pmatrix} \frac{1}{1-e_{(15)} t} & \frac{e_{(14)} t}{1-e_{(24)} t} \\ 0 & \frac{1}{1-e_{(24)} t} \end{pmatrix} \\
U_5(z, q, t) &= \begin{pmatrix} \frac{e_{(15)} t}{1-e_{(25)} t} & 0 \\ \frac{1}{1-e_{(25)} t} & \frac{1}{1-e_{(34)} t} \end{pmatrix} & U_6(z, q, t) &= \begin{pmatrix} \frac{1}{1-e_{(35)} t} & 0 \\ \frac{e_{(34)} t}{1-e_{(35)} t} & \frac{1}{1-e_{(5)} t} \end{pmatrix} \\
U_7(z, q, t) &= \begin{pmatrix} \frac{1}{1-e_{(45)} t} & \frac{1}{1-e_{(4)} t} \\ 0 & \frac{e_{(5)} t}{1-e_{(4)} t} \end{pmatrix} & U_8(z, q, t) &= \begin{pmatrix} \frac{1}{1-e_{(0)} q t} & \frac{e_{(45)} t}{1-e_{(3)} t} \\ 0 & \frac{1}{1-e_{(3)} t} \end{pmatrix}
\end{aligned} \tag{67}$$

Let us write a formula for $A_{\hat{\alpha}}^{\hat{\delta}}(z, q, t)$.

Suppose $\hat{\alpha}, \hat{\delta}$ are from the first column in the table (49). We choose $\hat{\delta}'$ to be complementary to $\hat{\delta}$. If we make a substitution (64) into iterated relation (66) we obtain

$$\begin{aligned}
A_{\hat{\alpha}}^{\hat{\delta}}(z, q, t) &= (A_{\hat{\alpha}}^{\hat{\delta}}(z, q, t) \ A_{\hat{\alpha}}^{\hat{\delta}'}(z, q, t)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \\
&= (A_{\hat{\alpha}}^{\hat{\alpha}}(z, q, t) \ 0) U_{\text{ht}(\hat{\alpha})} \cdots U_{\text{ht}(\hat{\delta})} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned}$$

We define a function

$$k(\hat{\alpha}) = \begin{cases} 0 & \text{if } \hat{\alpha} \text{ is in the first column of table (49)} \\ 1 & \text{otherwise} \end{cases}$$

Suppose $\hat{\alpha} = \alpha^r$. Here is a general formula for the character:

$$A_{\hat{\alpha}}^{\hat{\delta}}(z, q, t) = \left(\frac{1}{1-e_{\alpha} q^r t} \ 0 \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{k(\hat{\alpha})} U_{\text{ht}(\hat{\alpha})} \cdots U_{\text{ht}(\hat{\delta})} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{k(\hat{\delta})} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{68}$$

In this equation normalization formula (46) was taken into account.

In full analogy with equation (66) we obtain a set of equations for varying

lower indices.

$$\begin{aligned}
(A_{(0)}^{\delta}(t) A_{(3)-1}^{\delta}(t)) &= (A_{(12)}^{\delta}(t) A_{(2)-1}^{\delta}(t)) \begin{pmatrix} \frac{1}{1-e_{(0)}t} & \frac{e_{(12)}t}{1-e_{(3)}q^{-1}t} \\ 0 & \frac{1}{1-e_{(3)}q^{-1}t} \end{pmatrix} \quad l=1 \\
(A_{(12)}^{\delta}(t) A_{(2)-1}^{\delta}(t)) &= (A_{(13)}^{\delta}(t) A_{(1)-1}^{\delta}(t)) \begin{pmatrix} \frac{1}{1-e_{(12)}t} & \frac{1}{1-e_{(2)}q^{-1}t} \\ 0 & \frac{e_{(1)}q^{-1}t}{1-e_{(2)}t} \end{pmatrix} \quad l=2 \\
(A_{(13)}^{\delta}(t) A_{(1)-1}^{\delta}(t)) &= (A_{(14)}^{\delta}(t) A_{(23)}^{\delta}(t)) \begin{pmatrix} \frac{1}{1-e_{(13)}t} & 0 \\ \frac{e_{(23)}t}{1-e_{(13)}t} & \frac{1}{1-e_{(1)}q^{-1}t} \end{pmatrix} \quad l=3 \\
(A_{(14)}^{\delta}(t) A_{(23)}^{\delta}(t)) &= (A_{(15)}^{\delta}(t) A_{(24)}^{\delta}(t)) \begin{pmatrix} \frac{e_{(15)}t}{1-e_{(14)}t} & 0 \\ \frac{1}{1-e_{(14)}t} & \frac{1}{1-e_{(23)}t} \end{pmatrix} \quad l=4 \\
(A_{(15)}^{\delta}(t) A_{(24)}^{\delta}(t)) &= (A_{(25)}^{\delta}(t) A_{(34)}^{\delta}(t)) \begin{pmatrix} \frac{1}{1-e_{(15)}t} & \frac{e_{(25)}t}{1-e_{(24)}t} \\ 0 & \frac{1}{1-e_{(24)}t} \end{pmatrix} \quad l=5 \\
(A_{(25)}^{\delta}(t) A_{(34)}^{\delta}(t)) &= (A_{(35)}^{\delta}(t) A_{(5)}^{\delta}(t)) \begin{pmatrix} \frac{1}{1-e_{(25)}t} & \frac{1}{1-e_{(34)}t} \\ 0 & \frac{e_{(5)}t}{1-e_{(34)}t} \end{pmatrix} \quad l=6 \\
(A_{(35)}^{\delta}(t) A_{(5)}^{\delta}(t)) &= (A_{(45)}^{\delta}(t) A_{(4)}^{\delta}(t)) \begin{pmatrix} \frac{1}{1-e_{(35)}t} & 0 \\ \frac{e_{(4)}t}{1-e_{(35)}t} & \frac{1}{1-e_{(5)}t} \end{pmatrix} \quad l=7 \\
(A_{(45)}^{\delta}(t) A_{(4)}^{\delta}(t)) &= (A_{(0)1}^{\delta}(t) A_{(3)}^{\delta}(t)) \begin{pmatrix} \frac{e_{(0)}qt}{1-e_{(45)}t} & 0 \\ \frac{1}{1-e_{(45)}t} & \frac{1}{1-e_{(4)}t} \end{pmatrix} \quad l=8
\end{aligned} \tag{69}$$

9.1 Special cases

The formulas for characters $A_{\delta}^{\delta'}(z, q, t)$ undergo significant simplification after specialization $z = 1, q = 1$. The best result can be obtained when we further restrict the range of δ' . We shall explore this presently.

The set of weights J (11) is of interest because it is possible to write a vary simple recursive relation between corresponding Poincaré series.

Proposition 37 *The following recursions hold:*

$$\begin{aligned}
A_{\hat{\gamma}}^{(5)^r}(t) &= \frac{(1 - e_{15}q^rt - e_{14}e_{23}q^{2r}t^2 + e_{15}e_{14}e_{23}q^{3r}t^3)}{(1 - e_{13}q^rt)(1 - e_{34}q^rt)(1 - e_{5}q^rt)(1 - e_{23}q^rt)} A_{\hat{\gamma}}^{(15)^r}(t) + \\
&\quad \frac{e_1q^{r-1}t}{(1 - e_{13}q^rt)(1 - e_{34}q^rt)(1 - e_{5}q^rt)(1 - e_{23}q^rt)} A_{\hat{\gamma}}^{(1)^{r-1}}(t) \quad \hat{\gamma} \leq (1)^{r-1} \\
A_{\hat{\gamma}}^{(15)^r}(t) &= \frac{(1 - e_1q^{r-1}t - e_2e_{12}q^{2r-1}t^2 + e_1e_2e_{12}q^{3r-2}t^3)}{(1 - e_{13}q^rt)(1 - e_{14}q^rt)(1 - e_{15}q^rt)(1 - e_{12}q^rt)} A_{\hat{\gamma}}^{(1)^{r-1}}(t) + \\
&\quad + \frac{e_0q^rt}{(1 - e_{13}q^rt)(1 - e_{14}q^rt)(1 - e_{15}q^rt)(1 - e_{12}q^rt)} A_{\hat{\gamma}}^{(0)^r}(t) \quad \hat{\gamma} \leq (0)^r
\end{aligned}$$

$$\begin{aligned}
A_{\hat{\gamma}}^{(1)^r}(t) &= \frac{(1 - e_0 q^{r+1} t - e_{45} e_4 q^{2r} t^2 + e_0 e_{45} e_4 q^{3r+1} t^3)}{(1 - e_3 q^r t)(1 - e_2 q^r t)(1 - e_1 q^r t)(1 - e_4 q^r t)} A_{\hat{\gamma}}^{(0)^{r+1}}(t) + \\
&+ \frac{e_5 q^r t}{(1 - e_3 q^r t)(1 - e_2 q^r t)(1 - e_1 q^r t)(1 - e_4 q^r t)} A_{\hat{\gamma}}^{(5)^r}(t) \quad \hat{\gamma} \leq (5)^r \\
A_{\hat{\gamma}}^{(0)^r}(t) &= \frac{(q^3 - q^{2+r} e_5 t - q^{2r+1} e_{34} t^2 e_{25} + q^{3r} e_{34} t^3 e_5 e_{25})}{(q - e_{35} q^r t)(q - e_{45} q^r t)(1 - e_0 q^r t)(q - e_{25} q^r t)} A_{\hat{\gamma}}^{(5)^{r-1}}(t) + \\
&+ \frac{q^{2+r} e_{15} t}{(q - e_{35} q^r t)(q - e_{45} q^r t)(1 - e_0 q^r t)(q - e_{25} q^r t)} A_{\hat{\gamma}}^{(15)^{r-1}}(t) \quad \hat{\gamma} \leq (15)^{r-1}
\end{aligned}$$

Proof. We give details on the proof in case of $A_{\hat{\delta}}^{(5)^r}(t)$ only. All other cases can be treated similarly so the proofs are omitted. Iteration of the formula (38) leads to

$$A_{\hat{\delta}}^{(5)^r}(t) = \frac{1}{(1 - e_{15} q^r t)(1 - e_{14} q^r t)} A_{\hat{\delta}}^{(13)^r}(t) \quad (70)$$

Formulas

$$\begin{aligned}
A_{\hat{\delta}}^{(13)^r}(t) &= \frac{1}{1 - e_{13} q^r t} A_{\hat{\delta}}^{(12)^r}(t) + \frac{e_2 q^{r-1} t}{1 - e_{13} q^r t} A_{\hat{\delta}}^{(2)^{r-1}}(t) \\
A_{\hat{\delta}}^{(12)^r}(t) &= \frac{e_0 q^r t}{1 - e_{12} q^r t} A_{\hat{\delta}}^{(0)^r}(t) + \frac{1}{1 - e_{12} q^r t} A_{\hat{\delta}}^{(2)^{r-1}}(t)
\end{aligned}$$

are the special cases of (39). Expressions

$$\begin{aligned}
A_{\hat{\delta}}^{(2)^{r-1}}(t) &= (1 - e_1 q^{r-1} t)(1 - e_2 q^{r-1} t) A_{\hat{\delta}}^{(1)^{r-1}}(t) \\
A_{\hat{\delta}}^{(2)^{r-1}}(t) &= (1 - e_1 q^{r-1} t) A_{\hat{\delta}}^{(1)^{r-1}}(t)
\end{aligned}$$

are iterations of (38) applied backwards. We obtain our result by making substitutions of the presented identities in the order they are written into (70). ■

Let us introduce notations: $B_{\hat{\gamma}}^r(t) = A_{\hat{\gamma}}^{\hat{\delta}_r}(1, 1, t)$, $\hat{\delta}_r \in J$.

Corollary 38 *Under above assumptions on labeling weights the following recursions hold:*

$$B_{\hat{\gamma}}^r(t) = \frac{1+t}{(1-t)^2} B_{\hat{\gamma}}^{r-1}(t) + \frac{t}{(1-t)^4} B_{\hat{\gamma}}^{r-2}(t) \quad \delta_{r-2} \geq \hat{\gamma}$$

Proof. This recursion is a specialization of formulas from Proposition 37. ■

It follows from results of Examples 16 and 17 that

$$B_{(0)}^0(t) = A_{(0)}^{(15)}(1, 1, t) = \frac{1}{(1-t)^5}$$

and

$$B_{(0)}^5(t) = A_{(0)}^{(5)}(1, 1, t) = \frac{1-t^2}{(1-t)^8}$$

9.2 Delannoy polynomials

Delannoy polynomials (cf. [3]) $D_n(t) = \sum_{k=0}^n D_{k,n-k} t^k$, $\deg D_n = n$ is a recursive defined system

$$D_n(t) = (1+t)D_{n-1}(t) + tD_{n-2}(t)$$

$$D_0 = 1, \quad D_1(t) = 1+t$$

Here is a first few polynomials:

$$D_0(t) = 1$$

$$D_3(t) = 1+t$$

$$D_2(t) = 1+3t+t^2$$

$$D_3(t) = 1+5t+5t^2+t^3$$

$$D_4(t) = 1+7t+13t^2+7t^3+t^4$$

$$D_5(t) = 1+9t+25t^2+25t^3+9t^4+t^5$$

$$D_6(t) = 1+11t+41t^2+63t^3+41t^4+11t^5+t^6$$

$$D_7(t) = 1+13t+61t^2+129t^3+129t^4+61t^5+13t^6+t^7$$

$$D_8(t) = 1+15t+85t^2+231t^3+321t^4+231t^5+85t^6+15t^7+t^8$$

$$D_9(t) = 1+17t+113t^2+377t^3+681t^4+681t^5+377t^6+113t^7+17t^8+t^9$$

$$D_{10}(t) = 1+19t+145t^2+575t^3+1289t^4+1683t^5+1289t^6+575t^7+145t^8+19t^9+t^{10}$$

$$D_{11}(t) = 1+21t+181t^2+833t^3+2241t^4+3653t^5+3653t^6+2241t^7+833t^8+181t^9+21t^{10}+t^{11}$$

...

Delannoy numbers form a triangle $\{D_{n,m}\}$. A number $D_{n,m}$ describes the number of paths from the southwest corner $(0,0)$ of a rectangular grid to the northeast corner (m,n) , using only single steps north, northeast, or east.

Let

$$B_r(t) = \frac{D_r(t)}{(1-t)^{5+2r}}$$

The functions B_r satisfy recurrence (12)

Corollary 39 *Numerator of $B_{(0)}^r(t)$ is a Delannoy polynomial $D_r(t)$, $r \geq 0$.*

Proof. Sequences $B_{(0)}^r(t)$ and $B_r(t)$ satisfy the same recurrence relation and initial conditions. ■

The formula for generating function

$$\sum_{r \geq 0} B_r(t) s^r = \frac{1}{(-1+t)(s-st-st^2+st^3+ts^2-1+4t-6t^2+4t^3-t^4)}$$

follows easily from the equation (12). Its verification is left as an exercise.

Appendix

A Some automorphisms of \hat{A}

Remark 40 *Recall that $\text{Pin}(10)$ -group, the universal cover of $\text{O}(10)$, is generated by elements $e \in W + W'$ in Clifford algebra $Cl(W + W')$, that satisfy*

$$e^2 = 1. \tag{71}$$

In the basis v_1, \dots, v_5 of W and the dual basis v_{1}, \dots, v_{5*} in W^* element $e_i(a) = av_i + a^{-1}v_{i*}$ satisfies (71). in addition $e_i(a)e_j(b) = -e_j(b)e_i(a)$. An element $e_i(a)e_j(b)$ belongs to $\text{Spin}(10) \subset \text{GL}(S)$.*

The group $\text{Spin}_{\mathbb{C}}(10) = \text{Spin}(10) \times \mathbb{C}^\times / (-1, -1)$ has a tautological spinorial representation. Its Lie algebra acts in S by operators $v_i v_j, v_i v_{j*}, v_{i*} v_{j*}, 1$ (see e.g. [12], p131).

Let $u \in \text{Spin}_{\mathbb{C}}(10)$ to be $u = e_2 e_3 e_4 e_5$. Equation 71 and anti-commutativity of e_i imply $u^2 = 1$.

Lemma 41 *We have the following table of s -action on basis elements:*

$$\begin{aligned} u(\theta_{(0)}) &= \theta_{(1)}; u(\theta_{(12)}) = -\theta_{(2)}; u(\theta_{(13)}) = \theta_{(3)}; u(\theta_{(14)}) = -\theta_{(4)} \\ u(\theta_{(15)}) &= \theta_{(5)}; u(\theta_{(23)}) = -\theta_{(45)}; u(\theta_{(24)}) = \theta_{(35)}; u(\theta_{(25)}) = -\theta_{(34)}. \end{aligned} \quad (72)$$

We give only a half of the formulas. The remaining half can be recovered from the condition $u^2 = 1$.

Proof. Direct check. ■

We see that u permutes weight spaces and induces a transformation of the set E , which we by abuse of notations denote by u .

Definition 42 *A invertible transformation $h : M \rightarrow M$ of a poset M is an anti-automorphism if $\alpha \leq \beta$ implies $h(\alpha) \geq h(\beta)$.*

Proposition 43 *The poset E has a anti-automorphism u . It is realized as a symmetry of the graph E (see Picture 16) with respect to its geometric center. This symmetry can be realized by a linear transformation $u \in \text{Spin}_{\mathbb{C}}(10)$, that satisfies $u\theta_{\alpha} = \pm\theta_{u(\alpha)}$, $\alpha \in E$. The operator $u \in \text{Spin}(10)$ defines an automorphism of algebra A .*

Proof. The first assertion follows from Lemma 41. The operator u belongs to $\text{Spin}_{\mathbb{C}}(10)$. The operator must preserve linear space spanned by relations (28) in P and defines an automorphism of algebra A . ■

The automorphism u acts on \hat{P} and preserves ideal of relations I . We denote by \hat{u} the composition of u with transformation $\lambda^{\alpha}(z) \rightarrow \lambda^{\alpha}(\frac{1}{z})$. The automorphism \hat{u} permutes weight spaces and defines a transformation of \hat{E} , which we also denote by \hat{u} .

Proposition 44 *Transformation \hat{u} defines an anti-automorphism \hat{E}*

$$\hat{u} : (\alpha)^l \rightarrow u(\alpha)^{-l}. \quad (73)$$

It comes from an automorphism \hat{u} of the pair $I \subset \hat{P}$.

Proof. Note that $\hat{u}E_r = E_{-r}$. By Lemma 22 and by virtue of identity $\hat{u}T\hat{u} = T^{-1}$, it suffice to verify axioms of an anti-automorphism for $\hat{\alpha}, \hat{\beta} \in [(0), (1)^1]$. We leave this as an exercise. ■

The shift operator T is continuous on \hat{E} and defines an automorphism of \hat{P} :

$$\lambda_\alpha(z) \rightarrow \frac{\lambda_\alpha(z)}{z}. \quad (74)$$

On quadrics $\Gamma^{\mathbf{s}^k} \in \hat{P}(8)$ with no restrictions on l, l' indices it acts by the formula $T\Gamma^{\mathbf{s}^k} = \Gamma^{\mathbf{s}^{k+2}}$. Thus T is an automorphism of the pair $I \subset \hat{P}$.

B Lists of obstructions

Below is a list of pairs of noncommutative obstructions in E (see Definition 8)

$$\begin{aligned} o_{(0)} &= \{((4), ((5)(45))), ((5), ((4)(45)))\} \\ o_{(12)} &= \{((45), ((5)(35))), ((35), ((5)(45)))\} \\ o_{(13)} &= \{((45), ((5)(25))), ((25), ((5)(45)))\} \\ o_{(23)} &= \{((45), ((5)(15))), ((15), ((5)(45)))\} \\ o_{(14)} &= \{((35), ((5)(25))), ((25), ((5)(35)))\} \\ o_{(24)} &= \{((35), ((5)(15))), ((15), ((5)(35)))\} \\ o_{(15)} &= \{((34), ((5)(25))), ((5), ((5)(35)))\} \\ o_{(34)} &= \{((25), ((5)(15))), ((15), ((5)(25)))\} \\ o_{(25)} &= \{((34), ((5)(15))), ((5), ((15)(34)))\} \\ o_{(5)} &= \{((15), ((25)(34))), ((25), ((15)(34)))\} \\ o_{(35)} &= \{((24), ((5)(15))), ((5), ((15)(24)))\} \\ o_{(4)} &= \{((24), ((15)(34))), ((34), ((15)(24)))\} \\ o_{(45)} &= \{((23), ((5)(15))), ((5), ((15)(23)))\} \\ o_{(3)} &= \{((23), ((15)(24))), ((34), ((15)(23)))\} \\ o_{(2)} &= \{((23), ((15)(24))), ((24), ((15)(23)))\} \\ o_{(1)} &= \{((14), ((15)(23))), ((15), ((14)(23)))\} \end{aligned} \quad (75)$$

The sets o_α carry a label α , a weight in a \mathfrak{so}_{10} -spinor representation. Expressions h_α (35) carry the same label. To make connection of o_α with h_α more explicitly we identify the later with an element in $P \otimes I$ by means of substitution $x_s \rightarrow \Gamma^s$. Then o_α appears as a collection indices of noncommutative obstructions in h_α .

A list of noncommutative obstruction in $[(0), (1)^1] \subset \hat{E}$, that are not in the preceding list and not in $[(0)^1, (1)^1]$ contains elements of the form $(\alpha^i, (\beta^j, \gamma^k))$. Indices $i, j, k \geq 0$ satisfy $i + j + k = l = 1$ or 2. The table below describes the $l = 1$ case

$$\begin{aligned}
o_{(1)^1} &= \{((15), ((5), (0)^1)), ((0)^1, ((15), (5)))\} \\
o_{(2)^1} &= \{((25), ((5), (0)^1)), ((0)^1, ((25), (5)))\} \\
o_{(3)^1} &= \{((35), ((5), (0)^1)), ((0)^1, ((35), (5)))\} \\
o_{(4)^1} &= \{((45), ((5), (0)^1)), ((0)^1, ((45), (5)))\} \\
o_{(4)^1} &= \{((45), ((4), (0)^1)), ((0)^1, ((45), (4)))\} \\
o_{(45)^1} &= \{((5), ((0)^1, (4))), ((4), ((0)^1, (5)))\} \\
o_{(35)^1} &= \{((5), ((0)^1, (3))), ((3), ((0)^1, (5)))\} \\
o_{(25)^1} &= \{((5), ((0)^1, (2))), ((2), ((0)^1, (5)))\} \\
o_{(15)^1} &= \{((5), ((0)^1, (1))), ((1), ((0)^1, (5)))\} \\
o_{(34)^1} &= \{((4), ((0)^1, (3))), ((3), ((0)^1, (4)))\} \\
o_{(24)^1} &= \{((4), ((0)^1, (2))), ((2), ((0)^1, (4)))\} \\
o_{(14)^1} &= \{((4), ((0)^1, (1))), ((1), ((0)^1, (4)))\} \\
o_{(23)^1} &= \{((3), ((0)^1, (2))), ((2), ((0)^1, (3)))\} \\
o_{(13)^1} &= \{((3), ((0)^1, (1))), ((1), ((0)^1, (3)))\} \\
o_{(12)^1} &= \{((2), ((0)^1, (1))), ((1), ((0)^1, (2)))\} \\
o_{(0)^1} &= \{((2), ((12)^1, (1))), ((1), ((12)^1, (3)))\}
\end{aligned} \tag{76}$$

Obstructions with $l = 2$ can be transformed by symmetries T and \hat{u} into obstructions with $l = 1$. Because of this, the above table is sufficient for our purposes.

References

- [1] Y. Aisaka and E. A. Arroyo. Hilbert space of curved $\beta\gamma$ systems on quadric cones. *JHEP*, 0808(052), 2008.
- [2] Y. Aisaka, E. A. Arroyo, N. Berkovits, and N. Nekrasov. Pure spinor partition function and the massive superstring spectrum, 2008. arXiv:0806.0584v1 [hep-th].
- [3] C. Banderier and S. Schwer. Why delannoy numbers? *Journal of Statistical Planning and Inference*, 135(1):40–54, 11 2005.
- [4] G. Bergman. The diamond lemma for rings theory. *Advances in Math.*, 29(2):178, 1978.
- [5] N. Berkovits. Covariant quantization of the superparticle using pure spinors. *J. High Energy Phys.*, 09(016), 2001.
- [6] N. Berkovits and N. Nekrasov. The character of pure spinors. *Lett.Math.Phys.*, 74:75–109, 2005.
- [7] R. Bezrukavnikov. Koszul Property and Frobenius Splitting of Schubert Varieties, 1995. alg-geom/9502021.
- [8] G. Birkhoff. *Lattice theory*, volume 25 of *Colloquium Publications*. Amer. Math. Soc., 3 edition, 1968.
- [9] R. Bögvad. Some homogeneous coordinate rings that are koszul algebras, 1995. arXiv:alg-geom/9501011v2.
- [10] A. Braverman. Spaces of quasi-maps into the flag varieties and their applications. In *Proc. ICM 2006, Madrid, Zurich: Eur. Math. Soc., 2006*, volume 66, pages 3–45, River Edge, NJ, 2006. World Sci. Publ.
- [11] E. Cartan. *The theory of spinors*. Dover Books on Advanced Mathematics. Dover Publications, Inc. New York, 1981. With a foreword by Raymond Streater. A reprint of the 1966 English translation.

- [12] C. Chevalley. *The algebraic theory of spinors and Clifford algebras*, volume 2 of *collected works*. Springer, 1997.
- [13] A. Corti and M. Reid. Weighted grassmannians. In *Algebraic geometry*, pages 141–163. Walter de Gruyter and Co., Berlin, 2002.
- [14] D. Eisenbud. Introduction to algebras with straightening laws. In *Ring theory and algebra III*, pages 243–268. Dekker, 1980.
- [15] B. Feigin, M. Finkelberg, A. Kuznetsov, and I. Mirkovic. Semi-infinite flags. II. Local and global intersection cohomology of quasimaps’ spaces,. *Amer. Math. Soc. Transl. Ser. 2.*, 194:113–148, 1999.
- [16] B Feigin and E. Frenkel. Affine Kac-Moody algebras and semi-infinite flag manifolds. *Commun. Math. Phys.*, 128:161–189, 1990.
- [17] J.E. Humphreys. *Reflection groups and Coxeter groups*. CUP, 1990.
- [18] J.-I. Igusa. A classification of spinors up to dimension twelve. *American Journal of Mathematics*, 92(4):997–1028, October 1970.
- [19] V. Lakshmibai and P. Littelmann. Richardson varieties and equivariant k-theory. *J. Algebra*, 260(1):230–260, 2003.
- [20] V. Lakshmibai and K.N. Raghavan. *Standard monomial theory.. Invariant theoretic approach*, volume 137 of *Encyclopaedia of Mathematical Sciences*. Springer, 2008.
- [21] P. Littelmann. A littlewood-richardson rule for symmetrizable kac-moody algebras. *Invent. Math.*, 116(1-3):329–346, 1994.
- [22] P. Littelmann. Paths and root operators in representation theory. *Ann. of Math. (2)*, 142(3):499–525, 1995.
- [23] P. Littelmann. A plactic algebra for semisimple lie algebras. *Adv. Math.*, 124, no. 2, 1996, pp. 312–331., 124(2):312–331, 1996.

- [24] J. McCleary. *A User's Guide to Spectral Sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2 edition, Nov 2000.
- [25] T. Mora. An introduction to commutative and noncommutative groebner bases. *Theoretical Computer Science*, 134:131–173, 1994.
- [26] M.V. Movshev. On quasimaps to quadrics, 2010. arXiv:1008.0804v1 [math.QA].
- [27] A. Polishchuk and L. Positselski. *Quadratic algebras*, volume 37 of *Univ. Lecture Ser.* Amer. Math. Soc., Providence, RI, 2005.
- [28] M.S. Ravi. Coordinate rings of G/P are koszul. *Journal of Algebra*, 177(2):367–371, 1995.
- [29] F. Sottile and B. Sturmfels. A sagbi basis for the quantum Grassmannian. *J. Pure and Appl. Alg.*, 158:347–366, 2001.
- [30] E. B Vinberg and A. L. Onishchik. *Seminar on Lie groups and algebraic groups*. Springer series in Soviet mathematics. Springer Verlag, Berlin, New York, 1990.